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Michell Trusses Study, Rice U. summer 2013- Summary

The Michell Truss problem concerns the nature of a structure, constructed of bars and cables, of least cost that will balance a given set of point forces. The present subcollection of modules resulted from work of two NSF-VIGRE groups at Rice University. VIGRE is a program of Vertically Integrated Grants for Research and Education in the Mathematical Sciences funded by the National Science Foundation. The first was from 2008-9 academic year research PFUG*, who produced the Connexion module Michell Trusses. The second group, consisting of 4 undergraduates and a professor, worked during June, July 2013. They studied in more detail various aspects of problems related to Michell trusses, made some slight corrections to the previous module under the name Derived copy of Michell Trusses and wrote five additional modules. *A PFUG is a group of Postdocs, Faculty, Undergraduates and Graduate students formed around the study of a common problem.

The connection modules in this collection are: M47207, M47208, M47209, M47210, M47223, M47228.

Derived copy of Michell Trusses

This report summarizes work done as part of the Michell Trusses PFUG under Rice University's VIGRE program. VIGRE is a program of Vertically Integrated Grants for Research and Education in the Mathematical Sciences under the direction of the National Science Foundation. A PFUG is a group of Postdocs, Faculty, Undergraduates and Graduate students formed around the study of a common problem. In this PFUG, we study cost minimizing trusses. We ask what is the structure, constructed of bars and cables, of least cost that will withstand a set of point forces. This work was studied in the Rice University VIGRE class MATH499 in the Spring and Fall of 2008 and by the 2013 Michell research group at Rice University

Introduction

The Michell Truss PFUG studies a variational problem posed by mechanical engineer Anthony George Maldon Michell in the early part of the last century; what configuration of bars and cables is needed to withstand a system of balanced point forces is most economical? Suppose that some collection of forces with their points of application (point forces) is given and you are asked to build a structure which can withstand these forces. Your choice of materials are bars and cables. Bars can withstand compression (forces parallel and pointing inward) but will break under extension (forces parallel and pointing outward). Cables can withstand extension but will buckle under compression. You may also choose the strength of the bar or cable; the strength is the largest compression or extension the bar or cable respectively can withstand. A structure built of such cables and bars will be called a truss.

The first question the PFUG has addressed is whether one can build such a truss at all. Notice that a bar or cable by itself can withstand only one set of point forces; that of two opposite forces applied to the endpoints of the bar or cable. These point forces have the property that if the points of application were attached to a stationary body, the center of mass would not accelerate and the body would not rotate. This property is called balanced. Since point forces can be linearly superimposed, a necessary condition that a set of point forces be withstood by a truss is that it be balanced. Below we show that being balanced is also a sufficient condition; if the set of point forces is balanced then there is a truss withstanding it.

Next, the PFUG formulated necessary conditions for a truss to be economical. The cost of a bar or cable is its strength times its length. The cost of a truss is defined to be the sum of the cost of bars and cables constituted by the truss. In general, there are many trusses which withstand a given a set of point forces. The question is, which is most economical, i.e. which truss costs the least? In general this is a very difficult question to answer because a minimizing sequence, which we know to exist from the first part, may converge in a larger class of measures than described here and it is difficult to formulate any class of perturbations of a truss. Below we describe one class of perturbations on trusses with corners which yields a surprising necessary condition for economy. A set of perturbations to be studied in the future are also described.

Note: The 2013 Michell group has made some minor corrections and additions to the original module created by the 2008 PFUG, but this module is still largely the work of the 2008 PFUG.

Notation

We will restrict ourselves to points and vectors in three dimensional Euclidean space, $\mathbb{R}^3$. A *point force* is a vector valued measure $v\delta(a)$ where v is a vector and a is a point in Euclidean space and $\delta(\cdot)$ is the Kronecker delta mass. If φ is a continuous vector field on Euclidean space we define

Equation:

$$(v\delta(a), \varphi) = \varphi(a) \cdot v$$

and extend this definition linearly to a sum of point forces. A beam B is a pair of distinct points a and b in Euclidean space with a weight $\omega \in \mathbb{R}$. We can associate with B a mass

Equation:

$$\text{Cost}(B) = |\omega||a - b|$$

and point forces field

Equation:

$$\delta B = \omega \frac{a - b}{|a - b|} (\delta(a) - \delta(b)).$$

δB corresponds to the reaction force of a cable if $\omega < 0$ and of a bar if $\omega > 0$ with endpoints at a and b . Note that $(\delta B, \varphi) = 0$ if φ is a constant vector field. To see this, note that

Equation:

$$\begin{aligned} (\delta B, \varphi) &= \left(\omega \frac{a - b}{|a - b|} \delta(a) - \omega \frac{a - b}{|a - b|} \delta(b), \varphi \right) \\ &= \varphi(a) \cdot \omega \frac{a - b}{|a - b|} - \varphi(b) \cdot \omega \frac{a - b}{|a - b|} \\ &= 0 \end{aligned}$$

because $\varphi(a) = \varphi(b)$ since φ is a constant vector field. We choose the notation δB to be consistent with the notion of first variation of mass, although in this case it differs by the sign of ω . A truss T is a finite collection of beams and we define $\text{Cost}(T)$ and δT by extending the definitions [\[link\]](#) and [\[link\]](#) linearly;

Equation:

$$\text{Cost}(T) = \sum_{B \in T} \text{Cost}(B)$$

and

Equation:

$$\delta T = \sum_{B \in T} \delta B.$$

If $\mathbf{f}$ is a point force field (a sum of point forces) and T is a truss, then T is said to *equilibrate* $\mathbf{f}$ if $\delta T = \mathbf{f}$ in the sense that

Equation:

$$(\delta T, \varphi) = (\mathbf{f}, \varphi)$$

for all continuous vector fields $\varphi \cdot \mathbf{f} = f_1 \delta(a_1) + \cdots + f_\mu \delta(a_\mu)$ is said to be *balanced* if

Equation:

$$\sum_{i=1}^{\mu} f_i = 0$$

Equation:

$$\sum_{i=1}^{\mu} f_i \times a_i = 0.$$

Existence of equilibrating trusses

One may easily check that δB is equilibrated if B is a beam and by linearity δT is equilibrated if T is a truss. The first natural question we deal with is the converse question; is every balanced point force field equal to δT for some truss T ? In other words, can any balanced point force field be equilibrated by a truss? This section answers this question in the affirmative. The sufficiency will follow from a proof by induction.

Lemma 1 Let $\mathbf{f} = f\delta(a) + g\delta(b)$ be balanced. Then $\mathbf{f}$ is equilibrated by a truss T .

Let T consist of the single beam (a, b) with $\omega = g \cdot (a - b)/|a - b|$. Then [\[link\]](#) implies $f = -g$ and then [\[link\]](#) implies that f_2 is parallel to $a - b$, because $f \times (a - b) = 0$.

$\omega = \frac{g \cdot (a-b)}{|a-b|} = \frac{-f \cdot (a-b)}{|a-b|} = \frac{|-f||a-b|}{|a-b|} = |f|$ because f is parallel to $a - b$. Therefore $f = |f| \frac{a-b}{|a-b|}$ because $\frac{a-b}{|a-b|}$ is a unit vector parallel to f . Thus $f = \omega \frac{a-b}{|a-b|}$. By definition

Equation:

$$\delta T = \omega \frac{a-b}{|a-b|} (\delta(a) - \delta(b)) = f\delta(a) - f\delta(b) = f\delta(a) + g\delta(b) = \mathbf{f}.$$

Clearly then [\[link\]](#) holds.

Lemma 2 Let $\mathbf{f} = f\delta(a) + g\delta(b) + h\delta(c)$ be balanced with a, b and c not lying on the same line. Then $\mathbf{f}$ is equilibrated by a truss T .

Without loss of generality, a, b , and c lie in the xy -plane and $a = 0$. f, g , and h must lie in the plane, because otherwise it would be impossible to balance the forces. Because a, b , and c do not all lie on the same line, two of the points, say b and c , are linearly independent; dotting [\[link\]](#) with b and then c implies that $f \cdot e_3 = g \cdot e_3 = h \cdot e_3 = 0$ where e_3 is the unit basis vector parallel with the z -axis. Hence f can be expressed as a linear combination of b and c

Equation:

$$f = \omega_b \frac{b}{|b|} + \omega_c \frac{c}{|c|}.$$

Consider the point force field $\mathbf{e} = e_b \delta(b) + e_c \delta(c)$ where $e_b = g + \omega_b \frac{b}{|b|}$ and $e_c = h + \omega_c \frac{c}{|c|}$. We claim $\mathbf{e}$ is balanced;

Equation:

$$e_b + e_c = g + h + \omega_b \frac{b}{|b|} + \omega_c \frac{c}{|c|} = f + g + h = 0.$$

and

Equation:

$$\begin{aligned} e_b \times b + e_c \times c &= \left(g + \omega_b \frac{b}{|b|} \right) \times b + \left(h + \omega_c \frac{c}{|c|} \right) \times c \\ &= g \times b + b \times \omega_b \frac{b}{|b|} + h \times c + c \times \omega_c \frac{c}{|c|} \\ &= g \times b + h \times c \\ &= 0 \end{aligned}$$

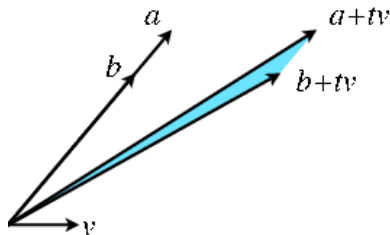
because b is parallel to $\omega_b \frac{b}{|b|}$ (and likewise for c) and $\mathbf{f}$ is balanced. According to Lemma 1, there is a truss R with equilibrating $\mathbf{e}$. Let S be the truss consisting of the collection of beams $(0, b)$ and $(0, c)$ with weights ω_b and ω_c respectively. We claim $T = R + S$ equilibrates $\mathbf{f}$. Let φ be a continuous vector field. Then

Equation:

$$\begin{aligned} \delta T &= \delta R + \delta S \\ &= \mathbf{e} + \omega_b \frac{b}{|b|} (\delta(b) - \delta(0)) + \omega_c \frac{c}{|c|} (\delta(c) - \delta(0)) \\ &= \left(g + \omega_b \frac{b}{|b|} \right) \delta(b) + \left(h + \omega_c \frac{c}{|c|} \right) \delta(c) + \omega_b \frac{b}{|b|} \delta(b) + \omega_c \frac{c}{|c|} \delta(c) - \delta(0) \left(\omega_b \frac{b}{|b|} + \omega_c \frac{c}{|c|} \right) \\ &= g\delta(b) - \omega_b \frac{b}{|b|} \delta(b) + h\delta(c) - \omega_c \frac{c}{|c|} \delta(c) + \omega_b \frac{b}{|b|} \delta(b) + \omega_c \frac{c}{|c|} \delta(c) - f\delta(0) \\ &= f\delta(a) + g\delta(b) + h\delta(c) \\ &= \mathbf{f} \end{aligned}$$

which then implies [\[link\]](#).

The following lemma shows when the volume of the parallelepiped spanned by three vectors can be made to be nonzero by sheering the parallelepiped in a fixed direction by an appropriate amount. An example in two dimensions is drawn below.



Lemma 3 Let a, b, c be distinct, nonzero points in Euclidean space and v a nonzero vector. Then there exists $t \in \mathbb{R}$ such that $v \in \text{Span}\{a + tv, b + tv, c + tv\}$.

If $v \in \text{Span}\{a, b, c\}$, then just take $t = 0$. If any of the vectors, say a and b , are linearly dependent, then for some constant, n , $b = na$. For any nonzero t , $a + tv$ is linearly independent from $b + tv$ because $b + tv = na + tv$ is not equal to $n(a + tv) = na + nt v$ since $nt v \neq tv$ because v and t are nonzero. Applying this argument again to b and c if necessary, we can get three vectors, $a + tv$, $b + tv$, and $c + tv$, which are guaranteed to be linearly independent. Because the span of any three linearly independent vectors is all of $\mathbb{R}^3$, every vector is contained in $\text{Span}\{a, b, c\}$, so it must be that $v \in \text{Span}\{a, b, c\}$, as desired.

Combining the above lemmas, we have

Theorem 1 A necessary and sufficient condition that a point force field $\mathbf{f} = f_1 \delta(a_1) + \cdots + f_\nu \delta(a_\nu)$ be balanced is that it is equilibrated by a truss T .

(Necessity) Suppose $\mathbf{f}$ is equilibrated by the truss T . Let $v \in \mathbb{R}^3$ and $\varphi(x) = v$ for all $x \in \mathbb{R}^3$. Then [\[link\]](#) implies

Equation:

$$0 = (\delta T, \varphi) = (\mathbf{f}, \varphi) = \sum_{i=1}^{\mu} f_i \cdot v = v \cdot \sum_{i=1}^{\mu} f_i.$$

because φ is a constant vector field. Since v was arbitrary, it can be chosen to be

Equation:

$$v = \sum_{i=1}^{\mu} f_i$$

so that

Equation:

$$0 = v \cdot \sum_{i=1}^{\mu} f_i = \sum_{i=1}^{\mu} f_i^2$$

and conclude that

Equation:

$$\sum_{i=1}^{\mu} f_i = 0$$

which implies [\[link\]](#). For $w \in \mathbb{R}^3$, define the skew symmetric matrix

Equation:

$$\widehat{w} = \begin{pmatrix} 0 & -w_3 & w_2 \\ w_3 & 0 & -w_1 \\ -w_2 & w_1 & 0 \end{pmatrix}$$

It is easy to check that $\widehat{w}\nu = w \times \nu$. Let $\nu \in \mathbb{R}^3$ and $\varphi(x) = \widehat{x}\nu$. Then

Equation:

$$\begin{aligned} (\delta T, \varphi) &= \sum_{i=1}^{\mu} (\delta B, \varphi) = \sum_{i=1}^{\mu} \omega_i (\varphi(a_i) - \varphi(b_i)) \cdot \frac{a_i - b_i}{|a_i - b_i|} = \sum_{i=1}^{\mu} \omega_i (\widehat{a_i} - \widehat{b_i})\nu \cdot \frac{a_i - b_i}{|a_i - b_i|} \\ &= \sum_{i=1}^{\mu} \omega_i (a_i - b_i) \times \nu \cdot \frac{a_i - b_i}{|a_i - b_i|} = 0. \end{aligned}$$

because the cross product $w_i(a_i - b_i) \times \nu$ is perpendicular to the vector $a_i - b_i$, which is parallel to $\frac{a_i - b_i}{|a_i - b_i|}$ so their dot product is zero. By [\[link\]](#), this implies

Equation:

$$0 = (\delta T, \varphi) = (\mathbf{f}, \varphi) = \sum_{i=1}^{\mu} f_i \cdot \varphi(a_i) = \sum_{i=1}^{\mu} f_i \cdot \widehat{a_i}\nu = \sum_{i=1}^{\mu} f_i \cdot a_i \times \nu = \sum_{i=1}^{\mu} f_i \times a_i \cdot \nu.$$

Since ν is arbitrary, we infer [\[link\]](#). Thus $\mathbf{f}$ is balanced.

(Sufficiency) We proceed by induction on ν . If $\nu = 2$, then $\mathbf{f}$ is equilibrated by the truss T guaranteed in Lemma 1. If $\nu = 3$ and a_1, a_2 and a_3 do not all lie on a line, then $\mathbf{f}$ is equilibrated by the truss T guaranteed in Lemma 2. If $\nu = 3$ and a_1, a_2 and a_3 lie on a line, consider the equivalent point force field $\mathbf{g} = f_1\delta(a_1) + f_2\delta(a_2) + f_3\delta(a_3) + 0\delta(a_4)$ where a_4 is a arbitrary point chosen off the line containing a_1, a_2 and a_3 . The result for $\nu = 4$ proves that $\mathbf{g}$ is equilibrated by a truss T . However, since $(\mathbf{g}, \varphi) = (\mathbf{f}, \varphi)$ for all continuous vector fields φ , this implies that T also equilibrates $\mathbf{f}$. Assume that sufficiency holds for $\nu \geq 3$. If $f_{\nu+1} = 0$, then we are done. Otherwise, according to Lemma 3, there is $\rho \in \mathbb{R}$ so that

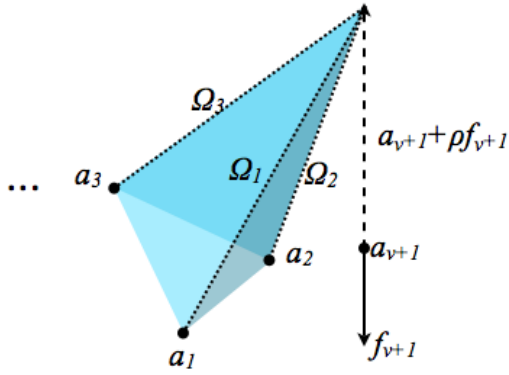
Equation:

$$f_{\nu+1} \in \text{Span}(a_1 - (a_{\nu+1} + \rho f_{\nu+1}), a_2 - (a_{\nu+1} + \rho f_{\nu+1}), a_3 - (a_{\nu+1} + \rho f_{\nu+1})).$$

Let $b = a_{\nu+1} + \rho f_{\nu+1}$ and let

Equation:

$$f_{\nu+1} = \Omega_1(a_1 - b) + \Omega_2(a_2 - b) + \Omega_3(a_3 - b).$$



Define the point force field $\mathbf{g} = g_1 \delta(b_1) + \cdots + g_\nu \delta(b_\nu)$ by

Equation:

$$\begin{aligned} g_i &= f_i + (a_i - b)\Omega_i, \quad i = 1, 2, 3, \\ g_i &= f_i, \quad i = 4, \dots, \nu, \quad b_i = a_i, \quad i = 1, \dots, \nu. \end{aligned}$$

We claim that $\mathbf{g}$ is balanced; by [\[link\]](#)

Equation:

$$\sum_{i=1}^{\nu} g_i = \sum_{i=1}^{\nu} f_i + \sum_{i=1}^3 (a_i - b)\Omega_i = \sum_{i=1}^{\nu} f_i + f_{\nu+1} = 0$$

and by [\[link\]](#)

Equation:

$$\begin{aligned} \sum_{i=1}^{\nu} g_i \times b_i &= \sum_{i=1}^{\nu} f_i \times a_i + \sum_{i=1}^3 \Omega_i (a_i - b) \times a_i \\ &= \sum_{i=1}^{\nu} f_i \times a_i + \sum_{i=1}^3 \Omega_i (a_i - b) \times (a_i - b) + \sum_{i=1}^3 \Omega_i (a_i - b) \times b \\ &= \sum_{i=1}^{\nu} f_i \times a_i + f_{\nu+1} \times (a_{\nu+1} + \rho f_{\nu+1}) = 0. \end{aligned}$$

By induction, there exists a truss R equilibrating $\mathbf{g}$. Let S be the truss consisting of the collection of beams $a_1 - b, a_2 - b, a_3 - b$ and $b - a_{\nu+1}$ with weights $\Omega_1, \Omega_2, \Omega_3$ and $|f_{\nu+1}|$ resp. Arguing as in Lemma 2, we find that $T = R + S$ equilibrates $\mathbf{f}$.

Some Economical Trusses

After proving the question of existence we turn to the question of economy. We say that a truss T is economical if it satisfies

Equation:

$$\text{Cost}(T) \leq \text{Cost}(S)$$

whenever $\delta S = \delta T$. That is to say, the cost of T is less than or equal to that of any truss which equilibrates the same force system as T . We begin by describing some global statements about economical trusses that can be proven by choosing special test vector fields φ in [\[link\]](#). Then we make local perturbations on trusses with corners to find a necessary condition for economy.

Some Global Properties

A surprising and easily proven fact is

Lemma 4 A truss is economical if the weights of the beams are all of the same sign. Such a truss lies in the closed convex hull of the support of the point forces it equilibrates.

To prove the first statement, note that

Equation:

$$\begin{aligned} (\delta T, x) &= \sum_{B \in T} (\delta B, x) = \sum_{B \in T} \omega \frac{a-b}{|a-b|} \cdot (a-b) = \sum_{B \in T} \omega |a-b| \\ &= \sum_{B \in T} \text{Cost}(B) = \text{Cost}(T). \end{aligned}$$

if $\omega > 0$ for each $B \in T$. On the other hand, if S is a truss with $\delta S = \delta T$, then

Equation:

$$\text{Cost}(T) = (\delta T, x) = (\delta S, x) = \sum_{B \in S} \omega |a-b| \leq \text{Cost}(S).$$

Hence T is economical.

To prove the latter statement, let K be the closure of the convex hull of the support of the point forces equilibrated by T . Then K is a convex polyhedron; let H be the hyper-plane passing through one of its sides. Without loss of generality, assume that H is the xy -plane and

$H_+ = \{x \in \mathbb{R}^3 : x_1 = x_2 = 0, x_3 > 0\}$ is the upper half space, and $\text{Support}(\delta T) \subset \mathbb{R}^3 \setminus H_+$. Let $\varphi(x) = e_3 \otimes e_3 x$ for $x_3 > 0$ and 0 otherwise where e_3 is the unit basis vector $(0, 0, 1)$ and $e_3 \otimes e_3$ is the tensor product

Equation:

$$e_3 \otimes e_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then,

Equation:

$$0 = (\delta T, \varphi) = \sum_{B \in T} \omega \frac{(a-b)}{|a-b|} \cdot e_3 \otimes e_3 (a-b) = \sum_{B \in T} \omega \frac{|a_3 - b_3|^2}{|a-b|}$$

which implies that $\omega = 0$ if it corresponds to a beam lying in H_+ .

Some Local Properties: Cutting Corners

Our object of interest are two dimensional trusses with corners. We have shown that an economical truss cannot have corners. To show this, we analyzed a perturbation of a truss with corners which consists of cutting the corner and replacing it with a flat top. Specifically, we showed that for any corner, the cut can be made sufficiently small so that the perturbed truss costs less. This surprising result suggests that any economical truss, if made of both cables and bars, has as boundary a differentiable curve and is supported on a set of positive two dimensional area.

We define a corner to be the union of three beams $\{B_1, B_2, B_3\} \subset T$ which share an endpoint p , lie in a halfplane about p and $(\delta B_1 + \delta B_2 + \delta B_3, \varphi) = 0$ for all φ with $\text{Support}(\varphi) \subset U$ for some neighborhood U of p .

By rescaling, translating and rotating, we may assume that B_1 and B_2 form two sides of an isosceles triangle and one endpoint of B_3 lies in the base of this triangle. The base of the triangle and B_1 form an angle of θ and the base of the triangle and B_3 form an angle of φ . The height of the triangle is l . We will need the relation

Equation:

$$\frac{\pi}{2} > \varphi > \theta > 0.$$

The sum of B_1, B_2 and B_3 will be called the four point truss, T_4 . The four point truss equilibrates the system of forces f_1, f_2 and f_3 with points of application the intersection of B_1, B_2 and B_3 resp. with the base of the triangle. We require

Equation:

$$f_1 + f_2 + f_3 = 0$$

or

Equation:

$$|f_1| + |f_2| = |f_3| \sin \varphi \csc \theta, \quad |f_1| - |f_2| = |f_3| \cos \varphi \sec \theta.$$

The simplicity of this truss allows an for a straight forward calculation of the cost. For $i = 1, 2, 3$ let ω_i and λ_i denote the strength and length resp. of beam B_i . It is easy to see that for this truss to equilibrate the the strengths of each of individual beam must be equal to the force that is based on a point of the beam and aligned to it, thus:

Equation:

$$\omega_1 = |f_1|, \quad \omega_2 = |f_2|, \quad \omega_3 = -|f_3|.$$

The lengths of the the T_4 are also easily calculated. B_1 and B_2 being the equal sides of an isosceles triangle it might be superfluous to state that:

Equation:

$$\lambda_1 = \lambda_2$$

With the use of trigonometry it is clear that

Equation:

$$\lambda_1 = \lambda_2 = l |\csc \theta|$$

and

Equation:

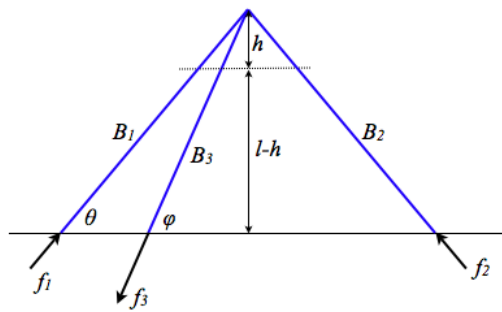
$$\lambda_3 = l |\csc \varphi|$$

The cost of T_4 is given by the formula

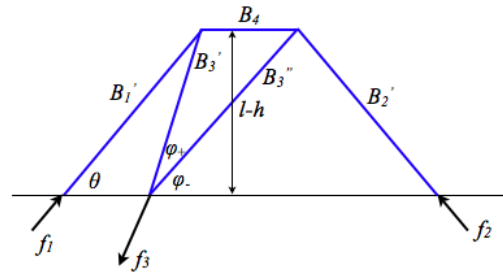
Equation:

$$\begin{aligned} \text{Cost}(T_4) &= \sum_{i=1}^3 |\omega_i| \lambda_i \\ &= |f_1| l \csc \theta + |f_2| l \csc \theta + |f_3| l \csc \varphi \\ &= l |f_3| (\sin \varphi \csc^2 \theta + \csc \varphi). \end{aligned}$$

We have used [\[link\]](#) in relating $|f_1|$ and $|f_2|$ to $|f_3|$. Note that the cost is linear in l and diverges as $\theta \rightarrow 0$ or $\varphi \rightarrow 0$.



Four Point Truss



Five Point Truss

Having calculated the cost of the T_4 we perturb the corner in order to produce a structure that may yield a lower cost compared to that of the T_4 . Our main objective being that of investigation of the consequences or eliminating corners from an already existing truss.

The modified truss will be obtained by removing the corner of the T_4 . A horizontal line parallel to the triangle base line a distance $l - h$ from the base of the isosceles triangle is added. The beams B'_1 and B'_2 are formed by shortening B_1 and B_2 to where they intersect this added line. To maintain a truss structure B_3 must be replaced by three new beams. B_4 connects B'_1 and B'_2 in the place where the cut of the corner was made. B'_3 shares a connections point with B_4 and B'_1 while B''_3 is connected to the node that is shared with both B_4 and B'_2 . The shape of the new truss that is obtained is that of a trapezoid, the applied forces remain same as for T_4 We shall hence forth refer to this truss structure and the five point tent truss, $T_{5,h}$.

The complicated nature of the $T_{5,h}$ makes the calculation of the lengths λ_i 's and strengths ω_i 's slightly less obvious than they were for the original T_4 . However, easy trigonometry and some calculations yield both the length's and the strengths of each individual beam. We use the same parameters to describe this truss as before, namely φ , θ and l . However, we need three more parameters to calculate the cost. These parameters are the following: h which describes the height of the cut and φ_+ and φ_- represent the angle between the base and B'_3 and B''_3 resp.

Keeping in mind some of the properties that govern the truss we have calculated the length's of each individual beams to be:

Equation:

$$\begin{aligned}\lambda'_1 &= (l - h) \csc \theta \\ \lambda'_2 &= (l - h) \csc \theta \\ \lambda'_3 &= \sqrt{(l \cot \varphi - h \cot \theta)^2 + (l - h)^2} \\ \lambda''_3 &= \sqrt{(l \cot \varphi + h \cot \theta)^2 + (l - h)^2} \\ \lambda_4 &= 2h \cot \theta\end{aligned}$$

The magnitude of strengths are similarly calculated to be:

Equation:

$$\begin{aligned}\omega'_1 &= |f_1| \\ \omega'_2 &= |f_2| \\ \omega'_3 &= |f_1| \sin \theta \csc \varphi_+ \\ \omega''_3 &= |f_2| \sin \theta \csc \varphi_- \\ |\omega_4| &= |f_1| \cos \theta + |f_2| \sin \theta \cot \varphi_+\end{aligned}$$

Having acquired all the information we require we can now calculate the cost of the $T_{5,h}$:

Equation:

$$\begin{aligned}
\text{Cost}(T_{5,h}) &= \sum_{i=1}^{\nu} \omega_i \lambda_i \\
&= |f_1| (l-h) \csc \theta + |f_2| (l-h) \csc \theta \\
&\quad + |f_1| \sin \theta \csc \varphi_+ \sqrt{(l \cot \varphi - h \cot \theta)^2 + (l-h)^2} \\
&\quad + |f_2| \sin \theta \csc \varphi_- \sqrt{(l \cot \varphi + h \cot \theta)^2 + (l-h)^2} \\
&\quad + |f_1| \cos \theta + |f_1| \sin \theta \cot \varphi_+ (2h \cot \theta)
\end{aligned}$$

In order to check that the perturbation lowers the cost for some sufficiently small h , we differentiate this expression and evaluate the derivate at $h = 0$. Even further simplification reduces the formula to the following somewhat complicated expression:

Equation:

$$\begin{aligned}
\frac{d}{dh} \text{Cost}(T_{5,h})_{h=0} &= |f_1| [-\csc \theta + \sin \theta \cos^2 \varphi (\cot \theta - \cot \varphi) \\
&\quad - \sin \theta (\cot \varphi \cot \theta + 1) + 2 \cot \theta (\cos \theta - \sin \theta \cot \varphi)] \\
&\quad + |f_2| [-\csc \theta - \sin \theta \cos^2 \varphi (\cot \varphi + \cot \theta) \\
&\quad + \sin \theta (\cot \varphi \cot \theta - 1)]
\end{aligned}$$

With the help of the above analysis, we have shown that

Equation:

$$\frac{d}{dh} \text{Cost}(T_{5,h})_{h=0} < 0$$

independent of l, θ and φ . This proves the claim that two dimensional trusses cannot have corners.

Other Deformations

Other deformations of trusses worth considering are perturbations of other junctions. Consider the case when $B_1, \dots, B_m$ are beams meeting at a point p . Suppose we shorten each beam by introducing a polyhedron about p whose m vertices meet the beams $B_1, \dots, B_m$ and each edge represents the location of a new beam. From Lemma 4 above we see that the cost does not change with the perturbation if the junction consists of only cables or only bars. This study would help rule out certain cases when both cables and bars meet at a junction.

A second deformation consists of making piecewise affine deformations within a triangle as follows; divide the triangle into three sub-triangles with common vertex in the center. Choose a base and let A_ρ be the affine transformation of $\mathbb{R}^2$ which maps points x to points $A_\rho(x)$ whose distance to the line passing through the base is scaled by ρ . The affine transformations in the remaining triangles is uniquely determined by continuity. Thus, $\eta_\rho(x)$ be the unique piecewise affine transformation of $\mathbb{R}^2$ with $\eta_\rho(x) = x$ for x not in the triangle and $\eta_\rho(x) = A_\rho(x)$ for x in the first sub-triangle. Applying η_ρ to a truss and then adding the necessary beams to ensure that is equilibrated then yields the perturbation.

Tensegrity

Tensegrity and Trusses

Introduction

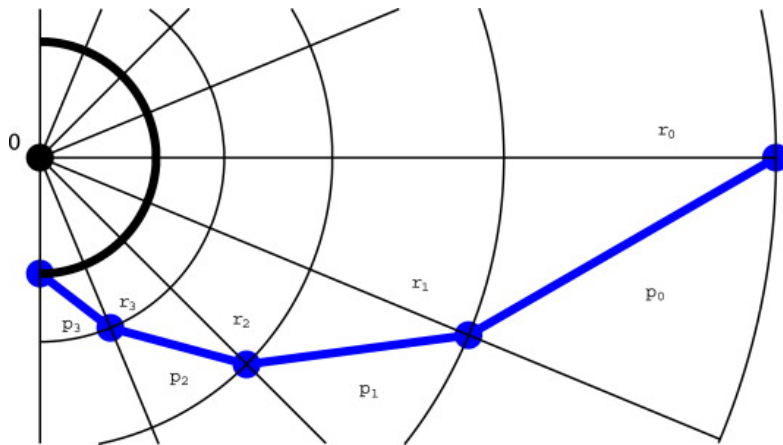
The term “tensegrity”, coined by Buckminster Fuller in the 1960s, is a combination of the words “tension” and “integrity”. In a tensegrity system, the compression members do not touch each other, and are stabilized by the tension members. Skelton and Oliveira, in their paper [\[link\]](#) define a *class-k tensegrity system* as “a tensegrity system such that no more than k rigid bodies make connections at a given node (with frictionless ball joints).”

Due to the high strength and low weight of tensegrity structures, Fuller imagined building “unprecedentedly large” structures [\[link\]](#), perhaps even large enough to cover a city. Kenneth Snelson, a former student of Fuller's, is renowned for his tensegrity sculptures, which are on display around the world. In addition to manufactured structures, tensegrity systems are abundant in nature. The elbow is a class 3 tensegrity joint, while the foot and shoulder are class 2 tensegrity joints. In a red blood cell, the protofilament is the compression member and the spectrin dimer are the tensile member [\[link\]](#).

This module further examines properties of tensegrity systems by looking at the paper “*Optimal tensegrity structures in bending: The discrete Michell truss*” by Robert E. Skelton and Mauricio C. de Oliveira. The following analysis and diagrams come largely from their paper, with changes made by the 2013 Michell group to help clarify their ideas for readers at the undergraduate level.

Michell Spirals

Consider a set of line segments, connected as shown below. In the diagram, $\varphi = \frac{\pi}{16}$ and $\beta = \frac{\pi}{6}$.



Definition 1 Let r_ℓ define a set of radii from a common origin, $\mathbf{0}$, for $\ell = 0, 1, 2, \dots, q$. Let p_ℓ , $\ell = 0, 1, 2, \dots, q - 1$ define the lengths of lines beginning at points with radius r_ℓ and terminating at points with radius $r_{\ell+1}$. Then a Michell spiral of order q is defined by the connections of lines of length p_ℓ , satisfying

Equation:

$$r_{\ell+1} = ar_\ell, \quad p_\ell = cr_\ell, \quad \ell = 1, 2, \dots, q,$$

where $a, c > 0$.

By a visual inspection, one can observe that

Equation:

$$r_{\ell+1} \cos \varphi + p_{\ell} \cos \beta = r_{\ell}$$

and

Equation:

$$r_{\ell+1} \sin \varphi = p_{\ell} \sin \beta$$

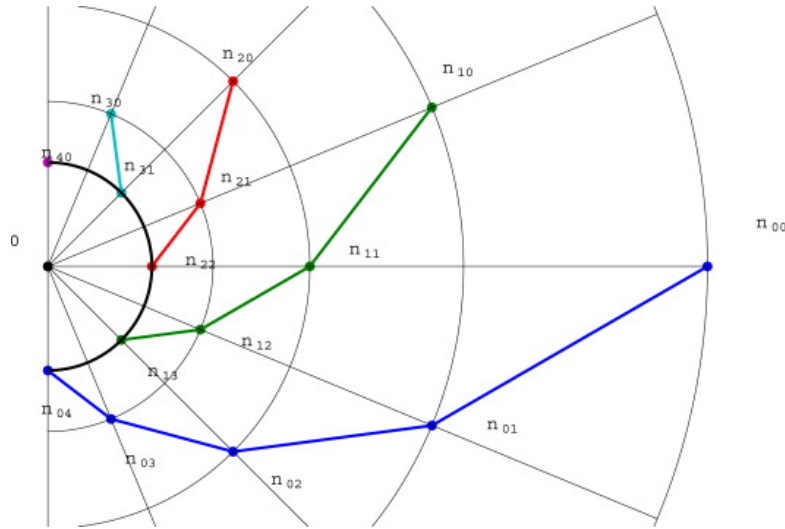
where φ is the angle between each radius and β is the angle between the beam and the radius.

Therefore, using some algebra and trigonometry, one can get that

Equation:

$$a = \frac{\sin \beta}{\sin (\beta + \varphi)} \quad \text{and} \quad c = \frac{\sin \varphi}{\sin (\beta + \varphi)}$$

In order for the spirals to converge to the origin, it must be that $a < 1$, which corresponds to $\varphi + 2\beta < \pi$.



Michell Topology

The magnitude of a node is determined by the radius on which it lies. That is,

Equation:

$$\| \mathbf{n}_{ik} \| = \| \mathbf{n}_{mn} \|$$

for all $i + k = m + n$.

Notice from the diagram above that nodes with the same radius are related by a phase-shift of $2m\varphi$, where m is an integer. Using complex notation, this can be expressed as

Equation:

$$\mathbf{n}_{i+m,k-m} = e^{j2m\varphi} \mathbf{n}_{ik}$$

The line segments comprising the spiral are described by defining the vector connecting nodes $\mathbf{n}_{ik}$ and $\mathbf{n}_{i,k+1}$,

Equation:

$$\mathbf{m}_{ik} = \mathbf{n}_{ik} - \mathbf{n}_{i,k+1}$$

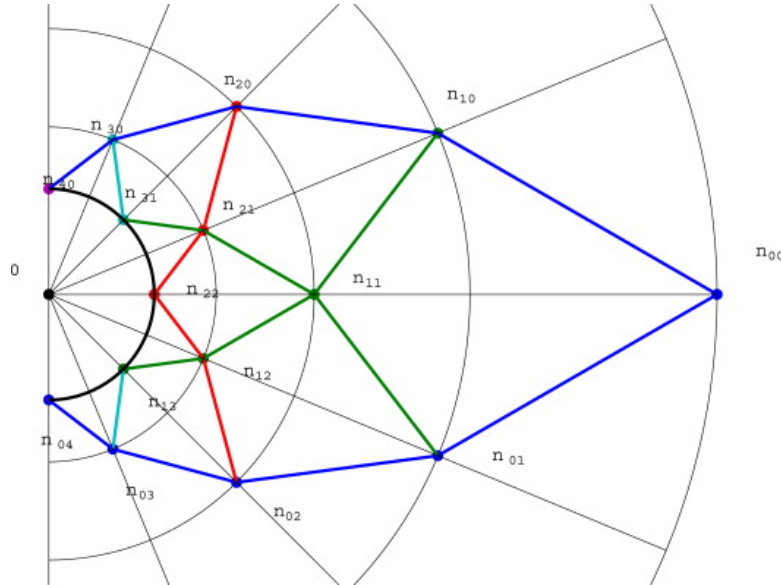
where the vector $\mathbf{n}_{ik}$ has magnitude and phase given by

Equation:

$$\mathbf{n}_{ik} = n_{ik} e^{j\varphi_{n_{ik}}}, \quad n_{ik} = r_{i+k}, \quad \varphi_{n_{ik}} = (i-k)\varphi$$

such that r_k satisfies the conditions in the definition for some specified r_0 . The mirror image of all lines reflected about the horizontal axis are obtained by computing the conjugate of the vectors $\mathbf{m}_{ik}$, and shown in [\[link\]](#) below.

Note that, by a visual inspection, $\mathbf{m}_{ik} = p_{i+k} e^{j[\beta + (i-k)\varphi]}$, where $p_m, m = 0, 1, 2, \dots, q$, satisfy [\[link\]](#).



Definition 2 A Michell Topology of order q consists of Michell spirals of order less than or equal to q and their conjugate spirals where [\[link\]](#) and [\[link\]](#) hold.

Force equilibrium at a generic node

A force applied to a node can be represented by the vector $\mathbf{w}_{ik}$ applied to the node $\mathbf{n}_{ik}$ and written in complex form as

Equation:

$$\mathbf{w}_{ik} = w_{ik} e^{j\varphi_{w_{ik}}}$$

where

Equation:

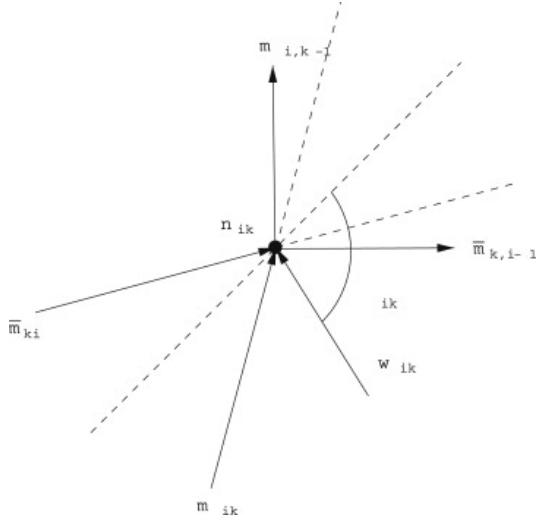
$$\varphi_{w_{ik}} = \theta_{ik} + (i - k)\varphi$$

and θ_{ik} is the angle at which the external force $\mathbf{w}_{ik}$ is applied, measured from the radial line to node $\mathbf{n}_k$.

At node $\mathbf{n}_{ik}$, the forces along directions $\mathbf{m}_{ik}$, $\bar{\mathbf{m}}_{ki}$, $\mathbf{m}_{i,k-1}$, $\bar{\mathbf{m}}_{k,i-1}$, respectively, have magnitudes f_{ik} , t_{ki} , $f_{i,k-1}$, $t_{k,i-1}$. Therefore, the sum of forces at node $\mathbf{n}_{ik}$ yields

Equation:

$$f_{ik} \frac{\mathbf{m}_{ik}}{\|\mathbf{m}_{ik}\|} + t_{ki} \frac{\bar{\mathbf{m}}_{ki}}{\|\mathbf{m}_{ki}\|} - f_{i,k-1} \frac{\mathbf{m}_{i,k-1}}{\|\mathbf{m}_{i,k-1}\|} - t_{k,i-1} \frac{\bar{\mathbf{m}}_{k,i-1}}{\|\mathbf{m}_{k,i-1}\|} + \mathbf{w}_{ik} = 0.$$



Multiplying [\[link\]](#) by the vector $p_{i+k} e^{j(k-i)\varphi}$ gives

Equation:

$$p_{i+k} \left[f_{ik} e^{j\beta} + t_{ki} e^{-j\beta} - f_{i,k-1} e^{j(\beta+\varphi)} - t_{k,i-1} e^{-j(\beta+\varphi)} + w_{ik} e^{j\theta_{ik}} \right] = 0.$$

Because both the real and imaginary parts of the above equation must be zero,

Equation:

$$\begin{bmatrix} \cos \beta & \cos \beta \\ -\sin \beta & \sin \beta \end{bmatrix} \begin{pmatrix} t_{ki} \\ f_{ik} \end{pmatrix} p_{i+k} - \begin{bmatrix} \cos (\beta + \varphi) & \cos (\beta + \varphi) \\ -\sin (\beta + \varphi) & \sin (\beta + \varphi) \end{bmatrix} \begin{pmatrix} t_{k,i-1} \\ f_{i,k-1} \end{pmatrix} p_{i+k} + \begin{bmatrix} \cos \theta_{ik} \\ \sin \theta_{ik} \end{bmatrix} w_{ik} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Linear propogation of forces

Define a vector $\mathbf{x}_\alpha \in \mathbb{R}^{2(\alpha+1)}$ which contains forces in all members that lie within the radii r_α and $r_{\alpha+1}$. That is,

$$\mathbf{x}_\alpha = \begin{bmatrix} t_{0,\alpha} \\ f_{\alpha,0} \\ t_{1,\alpha-1} \\ f_{\alpha-1,1} \\ \vdots \\ t_{i,\alpha-i} \\ f_{\alpha-1,i} \\ \vdots \\ t_{\alpha,0} \\ f_{0,\alpha} \end{bmatrix}$$

Define a vector $\mathbf{u}_\alpha$ such that

$$\mathbf{u}_\alpha = \begin{bmatrix} w_{\alpha+1,0} \\ w_{\alpha,1} \\ w_{\alpha-1,2} \\ w_{\alpha-2,3} \\ w_{\alpha-3,4} \\ \vdots \\ w_{0,\alpha+1} \end{bmatrix}$$

Theorem 1 Let a truss be arranged according to the Michell topology of order q , satisfying the conditions in the definitions, having external forces $\mathbf{w}_{ik}$ applied at the nodes $\mathbf{n}_{ik}$ with $i, k \geq 0$ and $i + k \leq q$. Let $\mathbf{x}_\alpha$ contain the forces(normalized by the member length) in all members within the band of members between radii r_α and $r_{\alpha+1}$. Then the forces propagate from one band to the next according to the linear recursive equation $\mathbf{x}_{\alpha+1} = \mathbf{A}_\alpha \mathbf{x}_\alpha + \mathbf{B}_\alpha \mathbf{u}_\alpha$ for some matrices $\mathbf{A}_\alpha$ and $\mathbf{B}_\alpha$.

Michell topologies under a single bending load

If $t_{ik}f_{ik} < 0$, then one member of the truss is in compression while the other is in tension, so the truss is said to be bending.

Theorem 2 Let a truss be arranged according to the Michell topology of order q satisfying [\[link\]](#) and with a and c given by [\[link\]](#). Assume the only force applied on the truss is $\mathbf{w} = we^{j\theta}$ at node $\mathbf{n}_{00}$. If

Equation:

$$\varphi > 0, \quad 0 < \beta < |\theta| < \pi - \beta, \quad \varphi + 2\beta < \pi,$$

then the forces at all nodes satisfy

Equation:

$$t_{ik}f_{ik} < 0 \quad \text{for all } i \leq q, \quad k \leq i$$

Furthermore, if

Equation:

$$\beta < \theta < \pi - \beta$$

then $t_{ik} > 0$ for all $i \leq q, k \leq i$. Otherwise, if

Equation:

$$\beta - \pi < \theta < -\beta$$

then $t_{ik} < 0$ for all $i \leq q, k \leq i$.

First, take the case where $q = 1$. In this case, $\mathbf{n}_{00}$ is the only node at which external forces are applied:

Equation:

$$\begin{pmatrix} t_{00} \\ f_{00} \end{pmatrix} = \frac{w_{00}}{\sin(2\beta)} \begin{bmatrix} \sin(\theta_{00} - \beta) \\ -\sin(\theta_{00} + \beta) \end{bmatrix}$$

In the case when $0 \leq \beta \leq \frac{\pi}{2}$, there are three cases of interest. The first is when $|\theta_{00}| < \beta < \frac{\pi}{2}$. Then $\sin(2\beta) \geq 0$ and

Equation:

$$-\pi < -2\beta < \theta_{00} - \eta < 0 \implies \sin(\theta_{00} - \beta) < 0$$

Equation:

$$0 < \theta_{00} + \beta < 2\beta < \pi \implies \sin(\theta_{00} + \beta) > 0$$

which implies $t_{00} < 0$ and $f_{00} < 0$, or, in other words, that both members of the truss are in tension.

The second case is when $|\theta_{00} - \pi| < \beta < \frac{\pi}{2}$. A similar analysis shows that $t_{00} > 0$ and $f_{00} > 0$ so both members of the truss are in compression.

Lastly, when $\beta < |\theta_{00}| < \pi - \beta$ then $t_{00}f_{00} < 0$, indicating that one member of the truss is in tension and the other in compression, so the truss is said to be bending.

Now, consider the case where $q > 1$ and assume that $0 < \beta < \theta_{00} < \pi - \beta$ so that $t_{00} > 0$ and $f_{00} < 0$ (the opposite case is handled similarly). Consider the node $\mathbf{n}_{01}$. In the force diagram, $\mathbf{w}_{01} = 0$ by assumption, hence $t_{1,-1} = 0$ because $\mathbf{n}_{01}$ is at the lower boundary of the truss. Repeat the analysis from before for the fictitious force $\mathbf{w}_{01} = |f_{00}|\mathbf{m}_{00}$. Because the truss has $a < 1$, $\beta < \theta_{01} = \varphi + \beta < \pi$, so

that $t_{10} > 0$ and $f_{01} < 0$. The same idea, applied to all nodes on the upper boundary with $k \geq q$, proves that $t_{k0} > 0$ and $f_{0k} < 0$ for all $k \leq q$.

The analysis at node $\mathbf{n}_{10}$ is similar. Repeat the above analysis for a fictitious force $\mathbf{w}_{10} = -|t_{00}|\overline{\mathbf{m}}_{00}$. The vector has a negative sign because the angle θ_{10} is measured clockwise, so the vector needs to be flipped to get the same angle θ as before. Therefore, for all nodes $\mathbf{n}_{i0}$ with $i \leq q$, $t_{0k} > 0$ and $f_{i0} < 0$ for all $k \leq q$.

Continuing on, for the node $\mathbf{n}_{11}$, use a fictitious force $\mathbf{w}_{11} = -|t_{01}|\overline{\mathbf{m}}_{01} + |f_{10}|\mathbf{m}_{10}$. Because $a < 1$, one can conclude that $\beta < \theta_{11} < \pi - \beta$ to get that $t_{11} > 0$ and $f_{11} < 0$, and continuing in this fashion gives $t_{ii} > 0$ and $f_{ii} < 0$ for all $i \leq q$.

The last type of node to consider is an interior node of the form $\mathbf{n}_{ik}$ where $i \neq k \neq 0$. At the node $\mathbf{n}_{ik}$ create the fictitious force $\mathbf{w}_{ik} = -|t_{i-1,k}|\overline{\mathbf{m}}_{20} + |f_{ii}|\mathbf{m}_{ii}$. The assumption that $a < 1$ leads to the conclusion that $\beta < \theta_{ik} < \pi - \beta$, yielding $t_{ki} > 0$ and $f_{ik} < 0$. Therefore, when $\beta < \theta_{00} < \pi - \beta$,
Equation:

$$t_{ki} > 0 \quad \text{and} \quad f_{ik} < 0 \quad \text{for all} \quad i, k \leq q$$

A similar procedure for $-\beta > \theta_{00} > \beta - \pi$ yields

Equation:

$$t_{ki} < 0 \quad \text{and} \quad f_{ik} > 0 \quad \text{for all} \quad i, k \leq q$$

Material volume of optimal structures

Let $\bar{\sigma}_j$, $j = 1, \dots, m$, denote the yield stress of the material used to construct the j th member, $\mathbf{m}_j$, loaded with force density σ_i and with constant cross section area A_j . To avoid material failure, the force density at any given member should satisfy

Equation:

$$\sigma_j(\|\mathbf{m}_j\|) \leq \bar{\sigma}_j \frac{A_j}{\|\mathbf{m}_j\|}, \quad j = 1, \dots, m$$

with equality holding in minimal volume structures. Some algebra yields

Equation:

$$v_j = \|\mathbf{m}_j\| A_j = \frac{1}{\bar{\sigma}_j} \sigma_j(\|\mathbf{m}_j\|) \|\mathbf{m}_j\|^2, \quad j = 1, \dots, m$$

where v_j is the volume of each member. Therefore, the total volume of a structure, V , is

Equation:

$$V = \sum_{j=1}^m v_j = \sum_{j=1}^m \frac{1}{\bar{\sigma}_j} \sigma_j(\|\mathbf{m}_j\|) \|\mathbf{m}_j\|^2.$$

Let b_j and s_j be the bar and string vectors for the j th compressive or tensile member in a structure with m_b bars and m_s strings. Therefore $V = V_b + V_s$, where

Equation:

$$V_b = \sum_{j=1}^{m_b} \frac{1}{\bar{\lambda}_j} \lambda_j (\| \mathbf{b}_j \|) \| \mathbf{b}_j \|^2 \quad \text{and} \quad V_s = \sum_{j=1}^{m_s} \frac{1}{\bar{\gamma}_j} \gamma_j (\| \mathbf{s}_j \|) \| \mathbf{s}_j \|^2$$

Assuming that all bars are the same material, $\bar{\lambda}_j = \bar{\lambda}$ for all $j = 1, \dots, m_b$. Likewise, if all strings are made of the same material, then $\bar{\gamma}_j = \bar{\gamma}$ for all $j = 1, \dots, m_s$. Hence

Equation:

$$V_b = \sum_{j=1}^{m_b} \frac{1}{\bar{\lambda}} \lambda_j (\| \mathbf{b}_j \|) \| \mathbf{b}_j \|^2 \quad \text{and} \quad V_s = \sum_{j=1}^{m_s} \frac{1}{\bar{\gamma}} \gamma_j (\| \mathbf{s}_j \|) \| \mathbf{s}_j \|^2$$

so that

Equation:

$$V = V_s + V_b = \sum_{j=1}^{m_b} \frac{1}{\bar{\lambda}_j} \lambda_j (\| \mathbf{b}_j \|) \| \mathbf{b}_j \|^2 + \sum_{j=1}^{m_s} \frac{1}{\bar{\gamma}_j} \gamma_j (\| \mathbf{s}_j \|) \| \mathbf{s}_j \|^2$$

Now, let J be defined by

Equation:

$$J = 2\bar{\lambda}\bar{\gamma}V + (\bar{\gamma} - \bar{\lambda}) \sum_{i=1}^n \mathbf{n}_i^T \mathbf{w}_i$$

From [\[link\]](#) and [\[link\]](#), it is true for a structure in static equilibrium that

Equation:

$$\sum_{i=1}^n \mathbf{n}_i^T \mathbf{w}_i = \sum_{j=1}^{m_s} \gamma_j (\| \mathbf{s}_j \|) \| \mathbf{s}_j \|^2 - \sum_{j=1}^{m_b} \lambda_j (\| \mathbf{b}_j \|) \| \mathbf{b}_j \|^2$$

Let $B = \sum_{j=1}^{m_b} \lambda_j (\| \mathbf{b}_j \|) \| \mathbf{b}_j \|^2$ and $S = \sum_{j=1}^{m_s} \gamma_j (\| \mathbf{s}_j \|) \| \mathbf{s}_j \|^2$ Therefore

Equation:

$$\begin{aligned} J &= 2\bar{\lambda}\bar{\gamma} \left[\frac{1}{\bar{\lambda}} B + \frac{1}{\bar{\gamma}} S \right] + (\bar{\gamma} - \bar{\lambda}) [S - B] \\ &= 2\bar{\gamma}B + 2\bar{\lambda}S + \bar{\gamma}S - \bar{\gamma}B - \bar{\lambda}S + \bar{\lambda}B \\ &= \bar{\gamma}B + \bar{\lambda}S + \bar{\gamma}S + \bar{\lambda}B \\ &= (\bar{\lambda} + \bar{\gamma}) (B + S) \end{aligned}$$

Thus

Equation:

$$J = \left(\bar{\lambda} + \bar{\gamma} \right) \left[\sum_{j=1}^{m_s} \gamma_j (\| \mathbf{s}_j \|) \| \mathbf{s}_j \|^2 + \sum_{j=1}^{m_b} \lambda_j (\| \mathbf{b}_j \|) \| \mathbf{b}_j \|^2 \right]$$

Assuming that $\bar{\lambda} = \bar{\gamma}$, we get that

Equation:

$$\begin{aligned} J &= 2\bar{\lambda}^2 V \\ &= 2\bar{\lambda}^2 \left[\sum_{j=1}^{m_b} \frac{1}{\bar{\lambda}} \lambda_j (\| \mathbf{b}_j \|) \| \mathbf{b}_j \|^2 + \sum_{j=1}^{m_s} \frac{1}{\bar{\lambda}} \gamma_j (\| \mathbf{s}_j \|) \| \mathbf{s}_j \|^2 \right] \\ &= 2\bar{\lambda} \left[\sum_{j=1}^{m_b} \lambda_j (\| \mathbf{b}_j \|) \| \mathbf{b}_j \|^2 + \sum_{j=1}^{m_s} \gamma_j (\| \mathbf{s}_j \|) \| \mathbf{s}_j \|^2 \right], \end{aligned}$$

which is consistent with [\[link\]](#). This implies that minimization of J is independent of the choice of material properties, and depends only on the variable topology.

The Steiner Tree Problem

An overview of the Steiner tree problem.

Introduction

Given a set of points, called terminals, a spanning tree is the set of terminals plus a number of edges, in which two terminals connected by exactly one simple path. The problem of the minimal spanning tree is to find the shortest spanning tree, where the length of a tree is the sum of the lengths of its edges. In some cases, points in addition to the original set, called Steiner points, can be carefully placed so that the spanning tree connecting the new Steiner points and the original set of points is shorter than any tree that could be created by adding, removing, or moving any Steiner point. The resulting spanning tree is called a Steiner tree.

The Michell truss problem begins with a set of point forces, which can be viewed as points in the Steiner tree problem. Because shorter beams cost less than longer beams of the same weight, it is beneficial to minimize the length of the beams used in a truss. Applications of the Steiner tree problem may help optimize a truss by minimizing the length of its beams.

Definition

The Toricelli point of a triangle, also called the Fermat point, is found by constructing two equilateral triangles on any three sides of the triangle, and drawing a line from the new vertex of each equilateral triangle to the opposite vertex of the original triangle. The location where these two lines intersect is the Toricelli point.

Each triangle has exactly one Toricelli point. In an acute triangle, the Toricelli point is inside the triangle, and the lines from the Toricelli point to each vertex of the triangle create 120° angles. Toricelli proved the shortest path connecting all three vertices of a triangle consists of three lines, each from the Toricelli point of the triangle to a vertex of the triangle.

Theorem 1

No two edges of a Steiner tree can meet at an angle less than 120° .

Proof

If the two edges meet at an angle of less than 120° , then select points on the edges that are equidistant from the vertex, and locate a new Steiner point at the Torricelli point of the triangle created by the vertex and the two selected points. Because the triangle is acute, the three new edges created by connecting the Toricelli point to the vertices of the triangle must be shorter in sum than the total length of the original two edges.

Theorem 2

A Steiner tree has no crossing edges.

Proof

If a Steiner tree had crossing edges, the cross would create at least two angles less than or equal to 90° .

Definition

The degree of a point in a tree is defined as the number of edges connected to that point.

Theorem 3

Each Steiner point of a Steiner tree is of degree 3.

Proof

From Theorem 1, no two edges meet at an angle of less than 120° , so the Steiner point can be of degree 3 at most. A Steiner point of degree 1 can simply be eliminated, as it creates an unnecessary edge, and a Steiner point of degree 2 can just be replaced with a straight line. Therefore a Steiner point must be of degree 3.

Theorem 4

A Steiner tree connecting n terminals contains at most $n - 2$ Steiner points.

Proof

The Euler characteristic for lines is 1, which means that $\# \text{vertices} - \# \text{edges} = 1$. Therefore there are $n + k - 1$ edges in a tree, where n is the number of terminal points and k is the number of Steiner points.

Each Steiner point has 3 edges coming from it, and each terminal point has at least one, so there are at least $\frac{3k+n}{2}$ total edges; the division by 2 accounts for the fact that each edge is counted at two vertices. Therefore

Equation:

$$n + k - 1 \geq \frac{3k + n}{2}$$

which, by simple algebra, yields

Equation:

$$n - 2 \geq k.$$

Definition

A Steiner tree with $n - 2$ Steiner points is a full Steiner tree.

Theorem 5

Every Steiner tree that is not a full Steiner tree can be decomposed into a union of full Steiner trees.

Proof

See [\[link\]](#)

Examples

In a 3 by 4 rectangle, two Steiner trees can be created, but the one with edge between the Steiner points parallel to the long side is the Steiner minimal tree.

In a 1 by 2 rectangle, the minimum spanning tree is not a Steiner tree.

Definition

A topology on a graph determines how many terminals and Steiner points there are, and how the points are connected to each other. A topology is a degeneracy of another if it can be obtained from shrinking edges of the original. The set of degeneracies of a topology G is denoted by $D(G)$.

A topology is a Steiner topology (full Steiner topology) if every Steiner point has degree 3 (and terminal has degree 1).

If F is a full Steiner topology, then $D_s(F)$ is the set of Steiner topologies in $D(F)$.

It is easy to see that a Steiner tree contains no closed shapes, called cycles. Otherwise, one edge could be deleted, and the resulting structure would be of shorter total length while still connecting all of the points.

Theorem 5

A Steiner tree whose topology is in $D(F)$ for some full Steiner topology F is the unique minimum tree among trees whose topologies are in $D_s(F)$.

Proof

Let T be a tree whose topology $G \in D_s(F)$. Let $v_1, \dots, v_n$ be an enumeration of the vertices. Define $f_{ij} = 1$ if $[v_i, v_j]$ is an edge in G and $f_{ij} = 0$ otherwise. Then the length of T is

Equation:

$$|T| = \sum_{i < j} f_{ij} |v_i - v_j|$$

$|v_i - v_j|$ is a norm, and norms are convex due to the triangle inequality, so T is strictly convex except when edges remain the same direction in a perturbation. In that case, when the three edges from a Steiner point are parallel, moving the Steiner point towards the side of the two overlapping edges decreases the total length, and thus $|T|$ is strictly convex. Because a Steiner tree is a local minimum, strict convexity guarantees that it is the unique minimum among trees with topologies in $D_s(F)$.

Corollary

There exists at most one relatively minimal tree for a given topology.

Definition

A Steiner hull for a given set of points N is a region which is known to contain a Steiner minimal tree.

In the Euclidean plane, the Steiner hull is bounded.

Lemma

Let $[v_i, v_j]$ be any path in a Steiner minimal tree, where v_i and v_j are defined as before. Let $L(v_i, v_j)$ be the region consisting of all points p satisfying

Equation:

$$|p - v_i| < |v_i - v_j| \text{ and } |p - v_j| < |v_i - v_j|$$

$L(v_i, v_j)$ is the lune-shaped intersection of circles of radius $|v_i - v_j|$ centered on v_i and v_j . No other vertex of the Steiner minimal tree can lie in $L(v_i, v_j)$.

Proof

If q were such a vertex, the Steiner minimal tree would contain either a edge from q to v_i not containing v_j , or vice versa. If the SMT contains a edge from q to v_i , for example, the SMT can be shortened by deleting $[v_i, v]$ and adding $[q, v_j]$, a contradiction.

Similar statements can be made to further specify where a Steiner hull may be.

Remark

It is easy to see that a full Steiner topology with n terminals has $2n - 3$ edges. Let $f(n)$, $n \geq 2$, denote the number of full Steiner topologies with $n - 2$ Steiner points. Adding a Steiner point to the middle of any of the edges and connecting it to a new terminal results in a full Steiner tree for $n + 1$ terminals. Geometrically, this causes the edge to become two separate edges, connected by the new Steiner point. In order to be consistent with the fact that edges must meet at 120° angles at Steiner points, the two new edges will rotate to meet at an angle of 120° with each other and with the edge connecting the new Steiner point to the new terminal. Thus there are $2n - 3$ ways to create a new full Steiner tree, so

Equation:

$$f(n + 1) = (2n - 3)f(n)$$

which has the solution

Equation:

$$f(n) = \frac{(2n - 4)!}{2^{n-2} (n - 2)!}$$

Let $F(n, k)$ denote the number of Steiner topologies with n terminals and k Steiner points such that no terminal is of degree 3. Then $F(n, k)$ can be obtained from $f(k)$ by first selecting $k + 2$ terminals and a full Steiner topology on it, and then adding the remaining $n - k - 2$ terminals one at a time at interior points of some edges, creating a "kink" in each edge. The first terminal can go to one of the $2k + 1$ edges, the second to one of $2k + 2$

edges, and so on, until the $(n - k - 2)$ th point is added to one of the $2k + (n - k - 2) = n + k - 2$ edges. Thus

Equation:

$$F(n, k) = \binom{n}{k+2} f(k) \frac{(n+k-2)!}{(2k)!}$$

Letting n_3 denote the number of terminals of degree 3

Equation:

$$F(n) = \sum_{k=0}^{n-2} \sum_{n_3=0}^{\frac{n-k-2}{2}} \binom{n}{n_3} F(n - n_3, k + n_3) \frac{(n + n_3)!}{k!}$$

A table containing the values of $f(n)$ and $F(n)$ for $n = 2, \dots, 8$ is given.

n	2	3	4	5	6	7	8
$f(n)$	1	1	3	15	105	945	10395
$F(n)$	1	4	31	360	5625	110880	2643795

Computational Complexity

The optimization problem:

Given: A set N of terminals in the Euclidean plane

Find: A Steiner tree of shortest length spanning N

can be recast as a decision problem:

Given: A set N of terminals in the Euclidean plane and an integer B

Decide: Is there a Steiner tree T that spans N such that $|T| \leq B$

The Steiner tree problem is NP-complete. This means that it is at least as hard as any problem in NP, but the solution is still verifiable in polynomial time. However, the problem cannot be solved in polynomial time. The discrete Euclidean Steiner problem, however, is not known to be NP-complete.

Given: A set N of terminals with integer coordinates in the Euclidean plane and an integer B .

Decide: Is there a Steiner tree T that spans N such that all Steiner points have integer coordinates and the discrete length of T is less than or equal to B , where the discrete length of each edge of T is the smallest integer not less than the length of that edge.

There is a version of this problem that is in P: Given a finite set in a Banach-Minkowski plane, a minimal Steiner tree for this set can be found in polynomially bounded time with an algorithm that executes only graph theoretic functions.

The Steiner Ratio

The Steiner ratio is the smallest possible ratio between the total length of a minimum Steiner tree and the total length of a minimum spanning tree. It is conjectured that, in the Euclidean Steiner problem, the Steiner ratio is $\frac{\sqrt{3}}{2}$. This is called the Gilbert-Pollak conjecture, and was thought to be proven in 1990, but some people do not accept this proof. Gilbert and Pollak also conjectured the ratio for higher dimensional spaces, but this was disproven.

It's easy to see that this is true in the four point case. Alternatively, the Steiner ratio is sometimes given as the total length of the minimum spanning tree over the length of the minimum Steiner tree. The Steiner ratio for the rectilinear Steiner tree, which is when all edges of the Steiner tree are perpendicular, is $\frac{3}{2}$.

Other applications

Steiner trees have many biological applications. They can be used to describe the ways proteins connect and fold. Steiner trees are also used in phylogeny. Organic chemistry diagrams often include Steiner trees. For example, CH_6N_3 is a regular hexagon Steiner tree.

Steiner trees have many computer science applications as well, such as in network design and circuit layout.

A Look at a Paper by Strang and Kohn

This module attempts to present the paper "Hencky-Prandtl Nets and Constrained Michell Trusses" by Gilbert Strang and Robert Kohn in a way that is accessible to undergraduate mathematics students.

Hencky-Prandtl Nets and Constrained Michell Trusses

Introduction

The paper "*Hencky-Prandtl Nets and Constrained Michell Trusses*" by Gilbert Strang and Robert Kohn [\[link\]](#) provides many interesting insights into the problem posed by Michell in his 1904 paper "*The limits of economy of material in framed-structures*" [\[link\]](#). This module attempts to present the former paper in a manner that can be understood by undergraduate students in mathematics.

Michell's problem assumes a material with given tensile and compressive yield limits, say $\pm\sigma_0$, and allows concentrated forces. A force F can only be carried by beams with cross-sectional area F/σ_0 , so if the points of application of the forces are too close together, there will be difficulty in finding enough space. Because there is no restriction on the magnitude or the spacing of the external loads, which are surface tractions f distributed along the boundary Γ of the given simply connected domain Ω , one needs to be careful to ensure Michell's problem is self-consistent. To do this, we remove the limitation imposed by σ_0 and allow increasingly strong beams whose cost is proportional to their strength. This way, the cross-sectional area doesn't get too large, so there isn't a problem with finding enough space.

Variational forms of the stress problems

The following is a brief derivation of the Michell problem, where the stress constraints come from the equilibrium equations:

Equation:

$$\operatorname{div} \sigma = 0 \text{ in } \Omega, \sigma : n = f \text{ on } \Gamma$$

where

Equation:

$$\sigma = \begin{bmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{bmatrix}$$

Equation:

$$\operatorname{div} \sigma = \begin{bmatrix} \frac{d}{dx} \sigma_x & \frac{d}{dy} \tau_{xy} \\ \frac{d}{dx} \tau_{xy} & \frac{d}{dy} \sigma_y \end{bmatrix}$$

and its eigenvalues, or principal stresses, are

Equation:

$$\lambda_{1,2} = \frac{1}{2} \left(\sigma_x + \sigma_y \pm \left((\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2 \right)^{1/2} \right)$$

The formula for [\[link\]](#) comes from the quadratic formula for the 2-D Lagrangian strain tensor. Bars are placed in the directions of principle stress, which, by definition, come from the orthogonal eigenvectors of σ . The required cross-sectional areas at any point are proportional (scaled by σ_0) to $|\lambda_1|$ and $|\lambda_2|$ and the total volume of the truss is proportional to

Equation:

$$\Phi(\sigma) = \int_{\Omega} (|\lambda_1| + |\lambda_2|) dx dy$$

Michell's optimal design can be determined from the solution to

Equation:

$$\begin{aligned} &\text{Minimize} && \Phi(\sigma) \\ &\text{subject to} && \text{div } \sigma = 0, \sigma : n = f \end{aligned}$$

To express the problem in a way that suggests a computational algorithm, we introduce a stress function $\psi(x, y)$. For any statically admissible stress tensor σ there is a function ψ such that

Equation:

$$\sigma = \begin{pmatrix} \psi_{yy} & -\psi_{xy} \\ -\psi_{xy} & \psi_{xx} \end{pmatrix}$$

where the subscripts indicate partial derivatives. It is easy to see that $\text{div } \sigma = 0$. In the first column, for example,

Equation:

$$\frac{\partial}{\partial x} \sigma_x + \frac{\partial}{\partial y} \tau_{xy} = \frac{\partial}{\partial x} \psi_{yy} + \frac{\partial}{\partial y} (-\psi_{xy}) = \psi_{yyx} - \psi_{xyy} = 0.$$

ψ is determined up to a linear function, because all second derivatives of a linear function are zero. Given the unit normal $n = (n_x, n_y)$, the unit tangent $t = (-n_y, n_x)$, and the condition

Equation:

$$\begin{pmatrix} \sigma_x & \tau_{xy} \\ \tau_{xy} & \sigma_y \end{pmatrix} \begin{pmatrix} n_x \\ n_y \end{pmatrix} = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix},$$

$\psi_{yy}n_x - \psi_{xy}n_y = f_1$ or $(\text{grad } \psi_y) \cdot t = f_1$ and similarly for f_2 . Therefore the tangential derivative of ψ_y is, by definition, $\frac{\partial \psi_y}{\partial t} = f_1$ or $\psi_y(P) = \int_{P_0}^P f_1 ds$ by the fundamental theorem of calculus, and similarly for f_2 . In order for the constraints to be compatible, the external loads f_1 and f_2 must be self-equilibrating, and therefore the boundary values ψ_x and ψ_y are well defined: the integrals around the closed curve Γ are zero. Provided that there is no net torque from the external forces, ψ itself can be determined. Write $\psi = g$, $\psi_n = h$, where ψ_n is the normal derivative of ψ .

If the signs of λ_1 and λ_2 are the same, then

Equation:

$$|\lambda_1| + |\lambda_2| = |\lambda_1 + \lambda_2| = |\psi_{xx} + \psi_{yy}|$$

because the trace of the matrix σ is equal to the sum of the eigenvalues.

If the λ_1 and λ_2 are of opposite signs, then

Equation:

$$\begin{aligned} |\lambda_1| + |\lambda_2| &= |\lambda_1 - \lambda_2| \\ &= \left| \frac{1}{2} \left(\sigma_x + \sigma_y \pm (\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2 \right)^{\frac{1}{2}} - \frac{1}{2} \left(\sigma_x + \sigma_y \mp (\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2 \right)^{\frac{1}{2}} \right| \\ &= \frac{1}{2} \left| \pm 2 \left((\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2 \right)^{\frac{1}{2}} \right| \\ &= \left((\sigma_x - \sigma_y)^2 + 4\tau_{xy}^2 \right)^{\frac{1}{2}} \end{aligned}$$

Therefore, in order to minimize the largest principle stress, the Michell problem becomes

Equation:

$$\begin{aligned} \text{Minimize} \quad & \int \int_{\Omega} \max(|\psi_{xx} + \psi_{yy}|, \left((\psi_{xx} - \psi_{yy})^2 + 4\psi_{xy}^2 \right)^{1/2}) dx dy \\ \text{subject to} \quad & \psi = g, \psi_n = h \text{ on } \Gamma \end{aligned}$$

In real life, unlike in Michell's problem, beams in a truss cannot be arbitrarily strong. Thus, if Λ denotes the upper bound on the stresses, the design problem for a constrained Michell truss is

Equation:

$$\begin{aligned} \text{Minimize} \quad & \int \int_{\Omega} |\lambda_1(\sigma)| + |\lambda_2(\sigma)| \\ \text{subject to} \quad & |\lambda_i| \leq \Lambda, \text{div } \sigma = 0, \sigma : n = f \text{ on } \Gamma \end{aligned}$$

The stress tensor is made up of two tensors: the mean hydrostatic stress tensor and the stress deviator tensor. The hydrostatic stress is the average of the normal stress components, σ_x and σ_y . The deviatoric stress is what's left over, namely $\sigma^D = \sigma - \frac{1}{2}(\sigma_x + \sigma_y)I$. The constraint on the strength of the beams can act only on σ^D instead of σ , therefore ignoring the hydrostatic component of the force. This is useful for working with a composite material that can only withstand pure pressure. The constraint then applies to the eigenvalues of σ^D . Using the same quadratic formula as in [\[link\]](#), we get that

Equation:

$$\left| \lambda_1(\sigma^D) \right|^2 = \left| \lambda_2(\sigma^D) \right|^2 = \left(\frac{1}{2}(\sigma_x - \sigma_y) \right)^2 + \tau_{xy}^2 \leq k^2$$

The matrix σ^D has trace zero, so the eigenvalues are of equal magnitude and opposite sign. Therefore the plain strain redesign problem is

Equation:

$$\begin{aligned} \text{Minimize} \quad & \iint_{\Omega} \left| \lambda_1(\sigma^D) \right| + \left| \lambda_2(\sigma^D) \right| \\ \text{subject to} \quad & \text{div } \sigma = 0 \text{ in } \Omega, \sigma : n = f \text{ on } \Gamma, \left| \lambda_1(\sigma^D) \right|^2 = \left| \lambda_2(\sigma^D) \right|^2 \leq k^2 \end{aligned}$$

Rewriting this in terms of the stress function ψ , and denoting it by $\|\psi\|$, we get

Equation:

$$\|\psi\|^2 = \left(\frac{1}{2}(\psi_{xx} - \psi_{yy}) \right)^2 + \psi_{xy}^2$$

Therefore the constrained problem above takes the form

Equation:

$$\begin{aligned} \text{Minimize} \quad & \iint_{\Omega} 2 \|\psi\| \, dx \, dy \\ \text{subject to} \quad & \|\psi\| \leq k \text{ in } \Omega, \psi = g, \psi_n = h \text{ on } \Gamma. \end{aligned}$$

If the k is too small, or f_1 and f_2 are too large, then the material may be too weak to withstand the load, and no truss can exist without collapsing.

The dual problems and optimality conditions

Now we give an informal derivation of the dual problem, which is a maximization over displacements $u = (u_1(x, y), u_2(x, y))$. No boundary conditions are imposed, because $\sigma : n = f$ was prescribed on all of Γ . To solve for the maximum over the constraint $\text{div } \sigma = 0$, we introduce a Lagrange multiplier. This is a system of two equations, $\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} = 0$ and $\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} = 0$, and we denote the multipliers by u_1 and u_2 . Then

Equation:

$$\begin{aligned} \min_{\substack{\text{div } \sigma = 0 \\ \sigma : n = f}} \int \int |\lambda_1| + |\lambda_2| &= \min_{\sigma : n = f} \max_u \int \int |\lambda_1| + |\lambda_2| + u \cdot \text{div } \sigma \\ &= \max_u \min_{\sigma} \int \int |\lambda_1| + |\lambda_2| - \langle \varepsilon(u), \sigma \rangle + \int_{\Gamma} u \cdot f \, ds \end{aligned}$$

where ε is the deformation tensor with components $\varepsilon_{ij} = \frac{1}{2} \left(\frac{\partial u_i}{\partial j} + \frac{\partial u_j}{\partial i} \right)$ and the inner product $\langle \varepsilon, \sigma \rangle$ is $\sum \sum \varepsilon_{ij} \sigma_{ij} = \text{trace } (\varepsilon \sigma)$.

The next step is minimization over σ , so ε remains fixed.

Equation:

$$\min_{\sigma} |\lambda_1(\sigma)| + |\lambda_2(\sigma)| - \langle \varepsilon, \sigma \rangle$$

First, we must diagonalize ε by a principle axis transformation, $\varepsilon = R^{-1} \varepsilon' R$ where $R^{-1} = R^t$, so that the eigenvalues ε_1 and ε_2 (the principal strains) appear in the diagonal matrix ε' . The inner product $\langle \varepsilon, \sigma \rangle$ and the eigenvalues $\lambda(\sigma)$ are invariant if this transformation is applied at the same time to σ :

Equation:

$$\langle \varepsilon, \sigma \rangle = \text{trace } (\varepsilon \sigma) = \text{trace } (R^{-1} \varepsilon' R R^{-1} \sigma' R) = \text{trace } (\varepsilon' \sigma') = \langle \varepsilon', \sigma' \rangle$$

Thus the minimization simplifies to

Equation:

$$\min |\lambda_1(\sigma')| + |\lambda_2(\sigma')| - \varepsilon_1 \sigma'_{11} - \varepsilon_2 \sigma'_{22}$$

If $|\varepsilon_1| > 1$, take $\sigma'_{11} = K \varepsilon_1$ and $\sigma'_{22} = 0$ for some $K \geq 0$ so that the expression now becomes $K |\varepsilon_1| - K |\varepsilon_1|^2 < 0$, and let $K \rightarrow \infty$ so that the minimum goes to $-\infty$. Similarly, the minimum is $-\infty$ if $|\varepsilon_2| > 1$. Otherwise, if $|\varepsilon_1|, |\varepsilon_2| \leq 1$, the minimum is zero. To see this, let $\sigma'_{ii} = 0$ if $|\varepsilon_i| < 1$ and $\sigma'_{ii} = K \varepsilon_i$ if $|\varepsilon_i| = 1$.

Substituting in this minimum, the dual problem is now

Equation:

$$\text{Maximize } \int_{\Gamma} (u_1 f_1 + u_2 f_2) \, ds \text{ subject to } |\varepsilon_i(u)| \leq 1$$

It's expected that this maximum value is equal to the minimum of [\[link\]](#), corresponding to the minimum volume of the truss.

Consider the case when $|\varepsilon_i| = 1$. When $\varepsilon_1 = 1$ and $\varepsilon_2 = -1$ the principal strains have equal magnitude and opposite sign, which can happen only for a special class of displacement fields. There is a condition $\frac{\partial^2 \theta}{\partial \alpha \partial \beta} = 0$ on the angle θ between the horizontal and the direction of the strain $\varepsilon_1 = 1$, and the secondary property $\varepsilon_1 + \varepsilon_2 = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} = 0$. Here, the geometrical problem is identical to that of slip lines in plane strain, where the eigenvalues of stress equal $\pm \sigma_0$. In the cases when $\varepsilon_1 = \varepsilon_2 = 1$ or $\varepsilon_1 = \varepsilon_2 = -1$ there is a similar correspondence with pure hydrostatic pressure in the stress case.

Because [\[link\]](#) is equal to 0, for every admissible σ and u ,

Equation:

$$\min_{\substack{\text{div } \sigma = 0 \\ \sigma : n = f}} \int_{\Omega} \left(|\lambda_1| + |\lambda_2| \right) dx dy = \max_u \int_{\Gamma} u \cdot f \, ds$$

so that

Equation:

$$\int_{\Omega} (|\lambda_1(\sigma)| + |\lambda_2(\sigma)|) \, dx \, dy \geq \int_{\Gamma} u \cdot f \, ds$$

with equality holding if the optimality condition is satisfied.

If σ and $\varepsilon(u)$ are simultaneously diagonal, the possibilities for σ can be restated as

Equation:

$$\begin{aligned} \text{if } \lambda_1(\sigma) &= 0, \text{ then } |\varepsilon_i(u)| \leq 1 \\ \text{if } \lambda_i(\sigma) &\neq 0, \text{ then } \varepsilon_i(u) = \text{sgn } \lambda_i \end{aligned}$$

because in a diagonal matrix, $\lambda_i(\sigma) = \sigma'_{ii}$.

The dual of the problem [\[link\]](#) becomes

Equation:

$$\min_{|\lambda_i| \leq \Lambda} \left| \lambda_1(\sigma) + |\lambda_2(\sigma)| - \langle \varepsilon, \sigma \rangle \right|$$

and the principal axis transformation leads to

Equation:

$$\min_{|\lambda_i| \leq \Lambda} \left| \lambda_1(\sigma') + |\lambda_2(\sigma')| - \varepsilon_1 \sigma'_{11} - \varepsilon_2 \sigma'_{22} \right|$$

Because λ_i cannot be arbitrarily large, $-\infty$ is no longer reached when $|\varepsilon_i| > 1$. Hence, there are now three cases, depending on the value of $|\varepsilon_i|$:

Equation:

$$\begin{aligned} \text{if } |\varepsilon_i| &> 1, \text{ then } \lambda_i = \Lambda \operatorname{sgn} \varepsilon_i \\ \text{if } |\varepsilon_i| &= 1, \text{ then } \lambda_i = |\lambda_i| \operatorname{sgn} \varepsilon_i, 0 \leq |\lambda_i| \leq \Lambda \\ \text{if } |\varepsilon_i| &< 1, \text{ then } \lambda_i = 0 \end{aligned}$$

In the second and third cases, the minimum value in [\[link\]](#) and [\[link\]](#) is still zero. However, in the first case, when $|\varepsilon_i| > 1$, the minimum is no longer $-\infty$. When σ' is diagonal with $\sigma'_{ii} = \lambda_i = \Lambda \operatorname{sgn} \varepsilon_i$, $|\lambda_i| - \varepsilon_i \sigma'_{ii} = \Lambda(1 - |\varepsilon_i|)$. Thus, the minimum in [\[link\]](#) and [\[link\]](#) can be expressed as

Equation:

$$M_\Lambda = \Lambda \min(0, 1 - |\varepsilon_1|) + \Lambda \min(0, 1 - |\varepsilon_2|)$$

Here, λ_i is being chosen through the minimum expression depending on the fixed ε_i . Hence the dual problem can now be stated as

Equation:

$$\text{Maximize } \int_{\Omega} M_\Lambda \, dx \, dy + \int_{\Gamma} (u_1 f_1 + u_2 f_2) \, ds$$

The cases of interest arise when one family of bars is in tension and the other compression, meaning $\lambda_1 \lambda_2 < 0$. Then there is a combination of Hencky-Prandtl nets, one coming from slip lines, and the other from Michell trusses.

The slip line net will occur when $\varepsilon_1 > 1$, $\varepsilon_2 < -1$, $\lambda_1 = \Lambda$, $\lambda_2 = -\Lambda$. In this case, $\|\psi\|$ represents the difference $\frac{1}{2}(\lambda_1 - \lambda_2)$, which has the constant value Λ , which is the yield surface in plane strain. The Michell situation occurs when $\varepsilon_1 = 1$, $\varepsilon_2 = -1$, $\lambda_1 = 0$, $\lambda_2 = 0$.

The other constrained problem [\[link\]](#) differs from the dual only in the condition of incompressibility:

Equation:

$$\begin{aligned} &\text{Maximize} && \int_{\Omega} M_k \, dx \, dy + \int_{\Lambda} (u_1 f_1 + u_2 f_2) \, ds \\ &\text{subject to} && \operatorname{div} u = 0 \end{aligned}$$

The kinematic constraint $\operatorname{div} u = 0$ arises because the trace of σ is unconstrained in [\[link\]](#). The optimality conditions relate the principal strains ε_i to the eigenvalues $\lambda_i^D = \lambda_i(\sigma^D) : \varepsilon$ and σ^D are simultaneously diagonal, and

Equation:

$$\begin{aligned} &\text{if } |\varepsilon| > 1, \text{ then } \lambda_i^D = k \operatorname{sgn} \varepsilon_i, \\ &\text{if } |\varepsilon| = 1, \text{ then } \lambda_i^D = |\lambda_i^D| \operatorname{sgn} \varepsilon_i, \quad 0 \leq |\lambda_i^D| \leq k \\ &\text{if } |\varepsilon| < 1, \text{ then } \lambda_i^D = 0. \end{aligned}$$

Open problems

- Find the most economical truss given a balanced point force field.
- Does the stress field in an optimal structure vary smoothly, so that the lines of principal stress are C^1 curves? The problem of regularity runs deeper than that of existence.

Michell Truss Stress Measure

More detailed and accessible derivations of the equations in section 2 of the paper "Michelle Trusses and Lines of Principal Action" are provided here. See Stress and Measure Theory modules for background material.

Michelle Trusses

Unidirectional Curve Stress

Let C be a simple curve in $\mathbb{R}^d$. In other words C is the image of a map $r \in C^{1,1}([0, 1], \mathbb{R}^d)$ such that $\dot{r}(t) \neq 0$ for each $t \in [0, 1]$ and r is injective. We will also require without loss of generality that $r(t)$ trace out the curve C at a constant speed, making $|\dot{r}(t)|$ constant on $[0, 1]$. Therefore $|\dot{r}(t)| = \frac{\int_0^1 |\dot{r}(s)| ds}{1-0} = \mathcal{H}^1(C)$ for each $t \in [0, 1]$ and the unit tangent vector to C at time t will be

Equation:

$$\tau(t) = \frac{\dot{r}(t)}{|\dot{r}(t)|} = \frac{\dot{r}(t)}{\mathcal{H}^1(C)}.$$

The curvature κ at time t is then given by

Equation:

$$\kappa(t) = \frac{d\tau/dt}{|dr/dt|} = \frac{\ddot{r}(t)/\mathcal{H}^1(C)}{\mathcal{H}^1(C)} = \frac{\ddot{r}(t)}{(\mathcal{H}^1(C))^2}.$$

We now define a measure σ^C that is proportional to the stress along the curve C .

Equation:

$$\sigma^C := (\tau \otimes \tau) \mathcal{H}^1 \Big|_C = \frac{1}{\mathcal{H}^1(C)^2} (\dot{r} \otimes \dot{r}) \mathcal{H}^1 \Big|_C$$

Thus σ^C is a rank 1 symmetric matrix measure.

We will now treat σ^C as the functional on $C_c(\mathbb{R}^d, \mathbb{R}^{d \times d})$ that corresponds to the measure σ^C . For any $\xi \in C_c(\mathbb{R}^d, \mathbb{R}^{d \times d})$,

Equation:

$$\begin{aligned}
\sigma^C [\xi] &= \int \langle \xi; \tau \otimes \tau \rangle d\mathcal{H}^1 \Big|_C \\
&= \frac{1}{\mathcal{H}^1(C)^2} \int_C \langle \xi(r); \dot{r} \otimes \dot{r} \rangle d|r| \\
&= \frac{1}{\mathcal{H}^1(C)} \int_0^1 \langle \xi(r(t)); \dot{r}(t) \otimes \dot{r}(t) \rangle dt.
\end{aligned}$$

To find the divergence of the measure σ^C , we let an arbitrary vector-valued function $u \in C_c^1(\mathbb{R}^d, \mathbb{R}^{d \times d})$ be given and we evaluate the functional $-\text{div } \sigma^C[u]$.

Equation:

$$\begin{aligned}
(-\text{div } \sigma^C)[u] &= \sigma^C[\nabla u] \\
&= \int_{\mathcal{R}^d} \nabla u d\sigma^C \\
&= \frac{1}{\mathcal{H}^1(C)} \int_0^1 \langle \nabla u(r(t)); \dot{r}(t) \otimes \dot{r}(t) \rangle dt
\end{aligned}$$

Using the definitions of the inner product and the tensor product, we can convert the matrix product $\langle \nabla u(r(t)); \dot{r}(t) \otimes \dot{r}(t) \rangle$ into a vector product:

Equation:

$$\begin{aligned}
\langle \nabla u(r(t)); \dot{r}(t) \otimes \dot{r}(t) \rangle &= \left\langle \nabla u(r(t)); \dot{r}(t) \dot{r}(t)^T \right\rangle \\
&= \text{tr} \left[\nabla u(r(t)) \left(\dot{r}(t) \dot{r}(t)^T \right)^T \right] \\
&= \text{tr} \left[\nabla u(r(t)) \dot{r}(t) \dot{r}(t)^T \right] \\
&= \langle \nabla u(r(t)) \dot{r}(t); \dot{r}(t) \rangle \\
&= \left\langle \frac{d}{dt} [u(r(t))]; \dot{r}(t) \right\rangle.
\end{aligned}$$

Therefore

Equation:

$$\begin{aligned}
(-\operatorname{div} \sigma^C)[u] &= \frac{1}{\mathcal{H}^1(C)} \int_0^1 \left\langle \frac{d}{dt}[u(r(t))]; \dot{r}(t) \right\rangle dt \\
&= \left\langle u(r(t)); \frac{\dot{r}(t)}{\mathcal{H}^1(C)} \right\rangle \Big|_{t=0}^1 - \frac{1}{\mathcal{H}^1(C)} \int_0^1 \langle u(r(t)); \ddot{r}(t) \rangle dt \\
&= \left\langle u(r(1)); \frac{\dot{r}(1)}{\mathcal{H}^1(C)} \right\rangle - \left\langle u(r(0)); \frac{\dot{r}(0)}{\mathcal{H}^1(C)} \right\rangle \\
&\quad - \frac{1}{\mathcal{H}^1(C)^2} \int_C \langle u(r); \ddot{r} \rangle dr \\
&= \int_{\mathcal{R}^d} \langle u; \tau \rangle d\delta_B - \int_{\mathcal{R}^d} \langle u; \tau \rangle d\delta_A - \int_{\mathcal{R}^d} \langle u; \kappa \rangle d\mathcal{H}^1 \upharpoonright_C \\
&= \tau\delta_B[u] - \tau\delta_A[u] - \kappa\mathcal{H}^1 \upharpoonright_C[u] \\
&= (\tau\delta_B - \tau\delta_A - \kappa\mathcal{H}^1 \upharpoonright_C)[u],
\end{aligned}$$

where $A = r(0)$ and $B = r(1)$. Since this equation holds for any $u \in C_c^1(\mathbb{R}^d, \mathbb{R}^{d \times d})$, we say that the measure $-\operatorname{div} \sigma^C$ is given by

Equation:

$$-\operatorname{div} \sigma^C = \tau\delta_B - \tau\delta_A - \kappa\mathcal{H}^1 \upharpoonright_C.$$

When the curve C is just equal to the line segment $[A, B]$, $\tau = \frac{B-A}{|B-A|}$ and $\kappa = 0$.

Therefore our equation becomes

Equation:

$$-\operatorname{div} \sigma^{[A,B]} = (\delta_B - \delta_A) \frac{B-A}{|B-A|}.$$

[\[link\]](#)

Stress in a Truss

The stress in a beam is given by $\frac{1}{2} \lambda \sigma^{[A,B]}$ for some λ . A finite collection of such beams is a *truss*. Therefore the stress in a truss that consists of l nodes at $\{A_1, A_2, \dots, A_l\}$ is given by

Equation:

$$\sigma = \sum_{m,n=1}^l -\lambda_{m,n} \sigma^{[A_m, A_n]},$$

where we require that $\lambda_{m,m} = 0$ for each m .

Recall that the stress σ of a structure that is in equilibrium must satisfy the restrictions $-\text{div } \sigma^T = F$ and $\sigma = \sigma^T$. It is easy to see that σ will be symmetric by definition. Therefore, if our truss is in equilibrium, there must be a force distribution F such that

Equation:

$$F = -\text{div } \sigma = -\text{div } \left(\sum_{m,n=1}^l -\lambda_{m,n} \sigma^{[A_m, A_n]} \right).$$

Since the operator $-\text{div}$ is linear, we get

Equation:

$$\begin{aligned} F &= \sum_{m,n=1}^l \lambda_{m,n} \text{div } \sigma^{[A_m, A_n]} \\ &= \sum_{m,n=1}^l \lambda_{m,n} (\delta_{A_m} - \delta_{A_n}) \frac{A_m - A_n}{|A_m - A_n|}, \end{aligned}$$

which will be our new requirement for a truss to be balanced [\[link\]](#).

Cost of a Truss

If we are given a force distribution F , a set of points $\mathcal{A} = \{A_1, A_2, \dots, A_l\}$, and a set of weights $\Lambda = \{\lambda_{m,n} : m, n = 1, \dots, l\}$ such that

$F = \sum_{m,n=1}^l \lambda_{m,n} (\delta_{A_m} - \delta_{A_n}) \frac{A_m - A_n}{|A_m - A_n|}$, then the cost of the truss structure will be the "total mass" of the structure, which is given by $\int_{\mathcal{R}^d} |\sigma|$.

Equation:

$$\begin{aligned}
\int_{\mathcal{R}^d} |\sigma| &= |\sigma|(\mathcal{R}^d) \\
&= |\sigma|\left(\bigcup_{1 \leq m < n \leq l} [A_m, A_n]\right) \\
&= \sum_{1 \leq m < n \leq l} |\sigma|([A_m, A_n]) \\
&= \sum_{1 \leq m < n \leq l} \left| \sum_{p,q=1}^l \lambda_{p,q} \sigma^{[A_p, A_q]} \right|([A_m, A_n]) \\
&= \sum_{1 \leq m < n \leq l} \left| \lambda_{m,n} \sigma^{[A_m, A_n]} + \lambda_{n,m} \sigma^{[A_n, A_m]} \right|([A_m, A_n]) \\
&= 2 \sum_{1 \leq m < n \leq l} \left| \lambda_{m,n} \frac{A_m - A_n}{|A_m - A_n|} \otimes \frac{A_m - A_n}{|A_m - A_n|} \mathcal{H}^1 \right|_{[A_m, A_n]}([A_m, A_n]) \\
&= \sum_{m,n=1}^l |\lambda_{m,n}| \frac{|(A_m - A_n) \otimes (A_m - A_n)|}{|A_m - A_n|} \left| \mathcal{H}^1 \right|_{[A_m, A_n]}([A_m, A_n]) \\
&= \sum_{m,n=1}^l \left| \lambda_{m,n} ||A_m - A_n| \right|
\end{aligned}$$

Cost Minimization

Given a truss T , which consists of a set of points $\mathcal{A} = \{A_1, A_2, \dots, A_l\}$ and a set of weights $\Lambda = \{\lambda_{m,n} : m, n = 1, \dots, l\}$, and given a force distribution F of finite support, we define a set $\Sigma_F^T(\overline{\Omega})$ of all admissible stress measures for the truss T .

Equation:

$$\Sigma_F^T(\overline{\Omega}) = \left\{ \sigma \in \mathcal{M}(\overline{\Omega}, S^{d \times d}) : -\operatorname{div} \sigma = F, \text{ and } \sigma = \sum_{m,n=1}^l -\lambda_{m,n} \sigma^{[A_m, A_n]} \right\}$$

The requirement $\sigma = \sum_{m,n=1}^l -\lambda_{m,n} \sigma^{[A_m, A_n]}$ is equivalent to the restriction that σ be rank 1 and supported on a finite collection of simple curves.

We also define the set

Equation:

$$\Sigma_F(\overline{\Omega}) = \left\{ \sigma \in \mathcal{M}(\overline{\Omega}, S^{d \times d}) : -\operatorname{div} \sigma = F \right\},$$

of all stress measures for structures that balance the force distribution F . The stresses in this set do not necessarily correspond to trusses, since we have dropped the requirement that the stress be rank 1 and concentrated on curves in $\mathbb{R}^d$. In fact, the stress measures in $\Sigma_F(\overline{\Omega})$ may be spread out over a region of $\mathbb{R}^d$. Also, we may allow the given force F to be diffuse.

Our original optimization problem was to find a truss T^0 such that the cost of T^0 is equal to $\inf_T \left\{ \inf \left\{ \int |\sigma| : \sigma \in \Sigma_F^T(\overline{\Omega}) \right\} \right\}$ for a given force distribution F of finite support. This is equivalent to finding a stress measure $\sigma^0 \in \bigcup_T \left(\Sigma_F^T(\overline{\Omega}) \right)$ such that

Equation:

$$\int |\sigma^0| = \inf \left\{ \int |\sigma| : \sigma \in \bigcup_T \left(\Sigma_F^T(\overline{\Omega}) \right) \right\}.$$

Such a σ^0 , however, does not always exist [\[link\]](#).

Tensors, Lagrange methods, and Hausdorff Measure

Presentation Notes Harry

The Stress Tensor:

Informally, tensors are geometric objects that describe linear relations between vectors, scalars, and other tensors (say, linear maps or dot products). A tensor can be represented by a multi-dimensional array of values and the number of indices needed to specify a component in this array is said to be the order of the tensor.

Multi-dimensional arrays:

1. Scalar $[a]$ - this is a tensor of order 0 because we don't need any indices to specify "a" within a 1×1 array
2. Vector $[a_i]_{i=1, \dots, n}$ - This is a tensor of order 1 because we need 1 index to specify a component of the vector
3. $m \times n$ matrix - this is a tensor of order 2
4. Tensors of higher order can be thought of as multidimensional "boxes"

Notice that we can create a tensor of a new order through the multiplication of tensors. For example, the products of a $n \times m \times p$ tensor with an $p \times 1$ vector will yield an $n \times m$ matrix.

Definition: The stress force on an object is said to be the force per unit area

So, the stress tensor is an order 2 tensor (a matrix) which takes a unit length direction vector normal to the surface of an object as an input and gives the stress force vector (with respect to the given coordinate system) at this surface as an output.

Then, for our problem, which is in 3 dimensions, the stress tensor is a 3×3 matrix with entries T_{ij} , where $i, j = 1, 2, 3$.

Imagine a beam and say that we would like to calculate the stress vector at a point, p , in the beam due to a force at the endpoint. Then, given a coordinate system, consider an "epsilon-cube" around p . Given an applied force at the endpoint of the beam (which translates to the same force at p) T_{ij} is the stress vector acting on the i th face in the $-j$ th direction - note that opposite faces have the same index. For an explanation of these properties, see [1].

One can check that in the context of a Michell Truss, the stress tensor will be a rank-one matrix and that the system of forces is in equilibrium if the divergence of the stress tensor is the negative of the applied forces. In other words, admissible trusses satisfy

Equation:

$$\nabla \sigma = -F$$

where F is the vector sum of the applied forces. See [5] for an explanation and derivation of this equation.

Definition: Strain is the amount of deformation an object experiences as a result of external forces when compared with its original size and shape.

The strain tensor operates similarly to the stress tensor, but the entries T_{ij} are replaced by partial derivatives of the position of a point, p , with respect to movement in the coordinate directions as a result of external forces. In the context of a Michell Truss, the strain tensor will always be a rank-one matrix as well.

One can investigate the relationship between stress and strain via a stress/strain curve.

Euler-Lagrange Equations

How do we optimize functionals? Consider a functional

Equation:

$$I(u) = \int_a^b f(x, u(x), u'(x)) dx.$$

over a region containing a and b.

Let $\mathbf{u}(\mathbf{x})$ be a minimizer for $I(u)$. Then $\mathbf{u}(\mathbf{x})$ should satisfy the Euler-Lagrange equations as follows:

Equation:

$$\frac{d}{dx} \left[\frac{df(x)}{d\mathbf{u}'(\mathbf{x})} (x, \mathbf{u}(\mathbf{x}), \mathbf{u}'(\mathbf{x})) \right] = \frac{df(x)}{d\mathbf{u}(\mathbf{x})} (x, \mathbf{u}(\mathbf{x}), \mathbf{u}'(\mathbf{x}))$$

For derivations and explanations of these equations, see [2] and [3].

Note that this doesn't mean that any function like $\mathbf{u}(\mathbf{x})$ satisfying the equations is a minimizer. In fact, any stationary function for $I(u)$ will satisfy the Euler-Lagrange equations. However, the point for us is that if we can create some sort of functional describing possible truss structures given a set of balanced forces, then we can employ the Euler-Lagrange equations to see which of these truss structures are "stationary," and thus possibly cost minimizers for supporting the given set of point forces.

The next question one might ask is how to construct such a functional for a finite set of point forces.

Using Lagrange Multipliers to Model a Finite System of Point Forces

The Lagrange multiplier method is used to find the maximum and minimum of a function subject to a set of constraints. For example, one could find the maximum and minimum values of the function $f(x,y) = 6x + 8y$ subject to the constraint $g(x,y) = x^2 + y^2 - 1 = 0$ (e.g. maximize and minimize the function on the unit circle). Our goal is to minimize the cost of an admissible truss structure subject to constraints regarding the application of a finite number of point forces. For example, given a balanced system of point forces in the plane,

Unknowns:

1. Weights of each possible beam in the structure with $w_{ij} = w_{ji}$
2. L_i multipliers to balance the x component of the applied forces
3. M_i multipliers to balance the y component of the applied forces

Constraints = balance of force components in each coordinate direction at each point of application. These equations are of the form:

At a point, a_0 :

(x-component force, y-component force) = L_i [sum of the x-components of the weights of each beam connected at a_0] + M_i [sum of the y-components of the weights of each beam connected at a_0]

As one might imagine, this problem can become complicated to solve in cases which lack symmetry. In the usual basic equilateral triangle example (with a point also at the centroid) that we used, there were still 4 constraint equations (one for each point of application) with an x-component part, y-component part, and all the unknown weights.

One should note that any solution for the simultaneous equations will generate an admissible truss structure. However, solving these equations simultaneously is a difficult problem in higher dimensions, or even when there are just more than four points and forces, none of which are symmetrically arranged. Overall, this is usually an impractical approach for finding a cost-minimizing truss structure given a set of balanced point forces.

Intro to Hausdorff Measure

Definition: If U is a nonempty subset of $\mathfrak{R}^n$, the diameter of U is said to be $|U| = \sup\{|x - y|, x, y \in U\}$

Definition: Fix $\delta > 0$. If $E \subset \bigcup U_i$ where $i = 1, \dots, \infty$ and $|U_i| \leq \delta$, we say that the collection of U_i 's is a δ -cover of E .

Definition: Let $E \subset \mathbb{R}^n$. For every $\delta > 0$ and $\alpha \geq 0$, $\delta, \alpha \in \mathfrak{R}$, let $H_\delta^\alpha(E) = \inf\{\sum |U_i|^\alpha, E \subset \cup U_i, |U_i| \leq \delta\}$ where $i = 1, \dots, \infty$. Then, we get the α -dimensional Hausdorff measure of E by letting δ go to 0 as we take the sum over all δ -covers.

So, $H^\alpha(E) = \lim_{\delta \rightarrow 0} H_\delta^\alpha(E) = \sup\{H_\delta^\alpha\}$ among all $\delta > 0$ since H_δ^α increases as δ decreases.

One important basic result in developing the notion of Hausdorff measure is that if δ is strictly less than the distance between two positively separated sets, E and F , then no set in a δ -cover of $E \cup F$ can intersect both E and F . This means that for some $\alpha > 0$,

$$H_\delta^\alpha(E \cup F) = H_\delta^\alpha(E) + H_\delta^\alpha(F)$$

Now, suppose we have two values for α , s and t , with $s < t$ and we want to consider the Hausdorff measure of a set $E \subset \mathbb{R}^n$. Then, $H_\delta^s(E) \geq \delta^s - t H_\delta^t(E) \Rightarrow$ if $H_\delta^t(E) > 0$, then $H_\delta^s(E)$ is infinite as δ goes to 0.

From this, we can see that there is a unique value for α , called the **Hausdorff dimension**, d , of E such that $H^\alpha(E) = \infty$ if $0 \leq \alpha < \dim E$ and $H^\alpha(E) = 0$ if $\dim E < \alpha < \infty$. For all of the above in the Hausdorff measure section, see [4].

So given a set E , how do we calculate its Hausdorff dimension?

Definition: Let $f: A \rightarrow B$ (and for our purposes, assume, $A, B \subset \mathbb{R}^n$). The function, f , is said to **Holder continuous** of exponent γ if: $\sup \frac{|f(x) - f(y)|}{|x - y|^\gamma} < \infty$ among all $x, y \in A$.

Proposition: If $f: A \rightarrow B$ is onto and Holder continuous for some γ , then $\dim(B) \leq \frac{1}{\gamma} \dim(A)$, where these are the Hausdorff dimensions of A and B .

Proof: Let $\cup U_i$, $i = 1, \dots, \infty$ be a δ -covering of A for some $\delta > 0$. Then, $f(\cup U_i)$ covers B and by the Holder continuity condition, we have that for each U_i , $C|A \cap U_i|^\gamma > |f(A \cap U_i)|$, where the bars denote the diameters of those sets and C is a constant.

This implies that for any α , $C^{\frac{\alpha}{\gamma}} |A \cap U_I|^\alpha > |f(A \cap U_i)|^{\frac{\alpha}{\gamma}}$

Taking this sum over all i , we get that $\sum |f(A \cap U_i)|^{\frac{\alpha}{\gamma}} < C^{\frac{\alpha}{\gamma}} \sum |A \cap U_i|^\alpha < C^{\frac{\alpha}{\gamma}} \sum |U_i|^\alpha \Rightarrow H^{\frac{\alpha}{\gamma}}(B) < C^{\frac{\alpha}{\gamma}} H^\alpha(A)$

Now, suppose the α is larger than $\dim(A)$. This means that $H^\alpha(A) = 0$ so that $H^\alpha(B) = 0$ as well. Decrease α until it reaches $\dim(A)$. Then, $\dim(B) \leq \frac{1}{\gamma} \dim A$, as desired.

We can use these measures in the context of the Michell problem. Developing the problem in new ways might allow one to use different techniques to find minimal cost structures among admissible configurations.

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